Sec 4.3 Linearly independent sets ; bases

Idea: Identify subsets that span a vector space H as "efficiently" as possible.

I. Def

Def Let V be a vector space.

The set $\{v_1, \dots, v_p\}$ in V is called ...

1) linearly independent if:

the equation $c_1 v_1 + c_2 v_2 + \dots + c_p v_p = 0$ (zero element in V)

has only the trivial solution $c_1 = 0, c_2 = 0, \dots, c_p = 0$.

2) linearly dependent if:

there are scalars $d_1, d_2, \dots, d_p$, not all zero, such that

$$d_1 v_1 + d_2 v_2 + \dots + d_p v_p = 0 \qquad (*)$$

In this case, the equation $(*)$ is called a linear dependence relation among the elements $v_1, v_2, \dots, v_p$.

Ex Recall $\mathbb{P} = \{\text{polynomials in } x\}$ forms a vector space.

Previously for Sec 4.1 lecture, we saw that we can write x^2 as a linear combination of 1, $1+x$, and $1+2x+x^2$:

$$x^2 = 1(1) + (-2)(1+x) + (1)(1+2x+x^2)$$

So $\underset{d_1}{(1)}(1) + \underset{d_2}{(-2)}(1+x) + \underset{d_3}{(1)}(1+2x+x^2) + \underset{d_4}{(-1)}x^2 = 0$

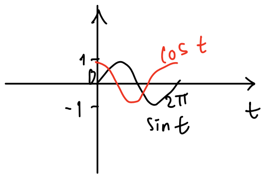
Therefore the set $\{1, 1+x, 1+2x+x^2, x^2\}$ is linearly dependent in $\mathbb{P}$.

Ex The set $C[0,2\pi]$ of all continuous functions on $0 \le t \le 2\pi$ forms a vector space.

The set $\{\sin t, \cos t\}$ is linearly independent in $C[0,2\pi]$

because $\sin t$ and $\cos t$ are not multiples of one another in $C[0, 2\pi]$,

i.e. there is no scalar c such that $c \cos t = \sin t$ for all t in $[0, 2\pi]$.



e.g. $\cos \frac{\pi}{2} = 0$ and $\sin \frac{\pi}{2} = 1$

and there is no scalar such that $c\,0 = 1$.

Def Let H be a subspace of a vector space V (possibly $H = V$).

A set $\mathcal{B}$ of elements in V is a basis for H if ...

1) $\mathcal{B}$ is a linearly independent set, AND

2) $H = \text{Span}\, \mathcal{B}$,

i.e. every element of the subspace H can be written as a linear combination of the elements in the set $\mathcal{B}$

Ex: Let $H = \mathbb{R}^n$, and let A be an $n \times n$ matrix

[The Invertible Matrix Theorem (Sec 2.3 pg 121) says:

A is invertible if and only if

the columns of A are linearly independent if and only if

the columns of A span $\mathbb{R}^n$

So, if A is invertible, then the columns of A are

(1) linearly independent and

(2) span $\mathbb{R}^n$,

and thus the columns of A form a basis for $\mathbb{R}^n$.

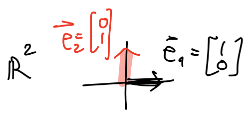
If A is non-invertible, the columns of A

are not linearly independent and also don't span $\mathbb{R}^n$.

Def Let $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ be the columns of the $n \times n$ identity matrix,

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad \vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

The set $\{\vec{e}_1, \dots, \vec{e}_n\}$ is called the standard basis for $\mathbb{R}^n$



Def/fact The set $\{1, t, t^2, \dots, t^n\}$ is a basis for $\mathbb{P}_n$, called the standard basis for $\mathbb{P}_n$

Ex: The standard basis for $\mathbb{P}_2$ is $\{1, t, t^2\}$.

II. Spanning Set Thm

Idea: Given a spanning set, we can construct a basis by discarding unneeded vectors

Thm Every spanning set for a vector space H contains a basis for H.

More specifically, let $S = \{v_1, \dots, v_p\}$ be a set in a vector space V, and let $H = \text{Span}\{v_1, \dots, v_p\}$.

a.) If one of the elements in S, say, v_k, is a linear combination of the remaining elements, then the set formed by removing v_k from S is still a spanning set of H.

b.) If $H \neq \{\text{zero element}\}$, then some subset of S is a basis for H.

III Bases for Nul A, Col A, and Row A

Find a basis for Nul A:

The method for finding the spanning set of Nul A in Sec 4.2 (see also the method for solving a linear system in Sec 1.5) produces a linearly independent set, and thus this spanning set is a basis for Nul A

Ex: Find a basis for Nul A, where $A = \begin{bmatrix} 1 & -2 & -1 & 3 \\ 2 & -4 & 1 & 0 \\ 1 & -2 & 2 & -3 \end{bmatrix}$

Write the augmented matrix of the system $A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ 2 & -4 & 1 & 0 & 0 \\ 1 & -2 & 2 & -3 & 0 \end{array}\right]$$

RREF

$$\xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This is the reduced echelon form

free variables

The general solution is $x_1 - 2x_2 + x_4 = 0$

$x_3 - 2x_4 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_2 - x_4 \\ x_2 \\ 2x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

the non-leading variables are the free variables

$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$ is a basis for Nul A. □

Find a basis for Col A:

Step 1: Put A into row echelon form (doesn't need to be reduced) B

Step 2: Check which columns of B have pivot positions.

Step 3: Keep only the columns of A which are pivot columns.

The result is a basis for Col A.

Ex Find a basis for Col A, where $A=\begin{bmatrix}1 & -2 & -1 & 3\\ 2 & -4 & 1 & 0\\ 1 & -2 & 2 & -3\end{bmatrix}$ as before

$$A=\begin{bmatrix}1 & -2 & -1 & 3\\ 2 & -4 & 1 & 0\\ 1 & -2 & 2 & -3\end{bmatrix} \xrightarrow{\text{Row reduce}} B=\begin{bmatrix}1 & -2 & -1 & 3\\ 0 & 0 & 3 & -6\\ 0 & 0 & 0 & 0\end{bmatrix}$$

This is in a (row) echelon form that's not reduced

pivot positions in columns 1 and 3

A basis for Col A is $\left\{ \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\2\end{bmatrix} \right\}$.

Find a basis for Row A:

Step 1: Put A into row echelon form (doesn't need to be reduced) B

Step 2: The nonzero rows of B form a basis for Row A.

Ex: A basis for Row A for the previous example is $\{ [1\ -2\ -1\ 3], [0\ 0\ 3\ -6] \}$

IV Views of a basis

A set $S = \{v_1, v_2, \ldots, v_n\}$ in a subspace H is ...

- ... a spanning set for H if every element of H can be written as a linear combination of S in at least one way (possibly many ways, making S too big)
- ... a linearly independent set if every element of H can be written as a linear combination of S in at most one way. (possibly no way, making S too small to be a spanning set)
- ... a basis for H if every element of H can be written as a linear combination of S in exactly one way. perfect!

Two views of basis:

1) A basis is a spanning set that is as small as possible.
2) A basis is a linearly independent set that is as big as possible.

Ex:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$$ spans $\mathbb{R}^3$ but is not linearly independent

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$$ is a basis for $\mathbb{R}^3$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\}$$ is linearly independent but does not span $\mathbb{R}^3$

Solutions to Group Quiz, pg 1

Q: True or false? The set of all solutions to $\vec{A}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is a subspace of $\mathbb{R}^4$

A: True, since this set is equal to the Nul A, which is a subspace of $\mathbb{R}^4$ by Thm 2.

Q: True or false? The set of all solutions to $\vec{A}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$ is a subspace of $\mathbb{R}^4$

A: False. This set does not contain the zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ in $\mathbb{R}^4$ since $\vec{A}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$, so this set is <u>not</u> a subspace.

Q: True or false? Let H be the set of all vectors in $\mathbb{R}^4$ whose coordinates a, b, c, d satisfy the equations $a - 2b + 5c = d$ and $c - a = b$. Then H is a subspace of $\mathbb{R}^4$.

A: True. Reasoning: H is the set of all solutions of

$$\left.\begin{aligned} a - 2b + 5c - d &= 0 \\ -a - b + c &= 0 \end{aligned}\right\}$$

which is a system of homogeneous linear equations. By Thm 2, H is a subspace of $\mathbb{R}^4$.

Q: True or false? Let H be the set of all vectors in $\mathbb{R}^4$ whose coordinates a, b, c, d satisfy the equations $a - 2b + 5c = d + 1$ and $c - a = b$. Then H is a subspace of $\mathbb{R}^4$.

A: False. Reasoning: H is the set of all solutions of

$$\left.\begin{aligned} a - 2b + 5c - d &= 1 \\ -a - b + c &= 0 \end{aligned}\right\}$$

which is a system of non-homogeneous linear equations. This is not a subspace because the zero vector is not in H.

Solutions to Group Quizzes pg 2

See also Sec 4.2 Exercise 44 (in MML)

Define a linear map $T: \mathbb{P}_2 \to \mathbb{R}^3$

by $T(p) = \begin{bmatrix} p(5) \\ p(5) \\ p(5) \end{bmatrix}$

a.) What is the kernel of T?

Find two polynomials in $\mathbb{P}_2$ that span the kernel of T

Sol: $\ker T = \{ p(x) \in \mathbb{P}_2 : p(5) = 0 \}$

$= \text{Span} \{ (t-5), (t-5)^2 \}$

b) What is the range of T?

Find a vector in $\mathbb{R}^2$ which spans the range of T

Sol: $\text{Range } T = \left\{ \begin{bmatrix} p(5) \\ p(5) \\ p(5) \end{bmatrix} : p(x) \in \mathbb{P}_2 \right\}$

$= \left\{ \begin{bmatrix} c \\ c \\ c \end{bmatrix} : c \in \mathbb{R} \right\}$

$= \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

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