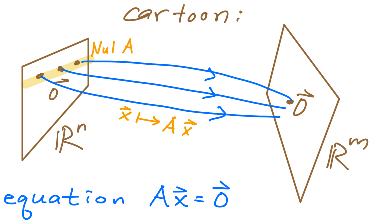
Sec 4.2 Null spaces, Column spaces, row spaces, & Linear Maps
I, II, III, IV

Part I Null space

cartoon:



Def The null space of an $m \times n$ matrix A is

$$\text{Nul } A = \{ \vec{x} : \vec{x} \text{ in } \mathbb{R}^n \text{ and } A\vec{x} = \vec{0} \},$$

↳ i.e. the set of all solutions of the homogeneous equation $A\vec{x} = \vec{0}$

Ex: $A = \begin{bmatrix} 1 & -2 & -1 & 3 \\ 2 & -4 & 1 & 0 \\ 1 & -2 & 2 & -3 \end{bmatrix}$ Q: Is $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ in Nul A ? Is $\vec{v} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ in Nul A ?

Ans: To test whether $\vec{u}$ and $\vec{v}$ satisfy $A\vec{x} = \vec{0}$, simply compute

$$A\vec{u} = \begin{bmatrix} 1 & -2 & -1 & 3 \\ 2 & -4 & 1 & 0 \\ 1 & -2 & 2 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1(2) - 2(1) - 1(0) + 3(0) \\ 2(2) - 4(1) + 0 + 0 \\ 1(2) - 2(1) + 0 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad A\vec{v} = \begin{bmatrix} 4 \\ 8 \\ 4 \end{bmatrix} \neq \text{zero vector}$$

So $\vec{u}$ is in Nul A, and $\vec{v}$ is not in Nul A. □

Thm 2 The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

↳ (Equivalently, the set of all solutions to a system $A\vec{x} = \vec{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.)

Proof: Nul A is a subset of $\mathbb{R}^n$ because A has n columns.
Show that Nul A satisfies all three properties of a subspace

1. (Containing the zero vector) $\vec{0} \in \mathbb{R}^n$ is in Nul A since $\underset{\text{in } \mathbb{R}^n}{A\vec{0}} = \underset{\text{in } \mathbb{R}^m}{\vec{0}}$

2. (Closed under vector addition) Suppose $\vec{u}$ and $\vec{v}$ are in Nul A.

Then $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$ (by distributivity property of matrix multp)

$= \vec{0} + \vec{0}$ since $\vec{u}$ and $\vec{v}$ are in Nul A

$= \vec{0}$

Thus $\vec{u} + \vec{v}$ is in Nul A.

3. (Closed under scalar multiplication) Suppose c is any number and $\vec{u}$ is in Nul A.

Then $A(c\vec{u}) = c(A\vec{u})$

$= c(\vec{0})$ since $\vec{u}$ is in Nul A

$= \vec{0}$

Thus $c\vec{u}$ is in Nul A. ▨

Ex. Find a spanning set of Nul A. (Review process in Sec 1.5)

The augmented matrix of the system $A\vec{x} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ is

Note: A is the same matrix I used in the notes from the first day of class

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ 2 & -4 & 1 & 0 & 0 \\ 1 & -2 & 2 & -3 & 0 \end{array}\right]$$

Row reduce:

$$\begin{array}{r} \\ -2\,\text{Row I} + \text{Row II} \\ -\text{Row I} + \text{Row III} \end{array} \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & -6 & 0 \\ 0 & 0 & 3 & -6 & 0 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -\text{Row II} + \text{Row III} \end{array} \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This is in a (row) echelon form

We can stop here or keep going to get the reduced echelon form

$$\begin{array}{r} \\ \tfrac{1}{3}\,\text{Row II} \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{r} \text{Row II} + \text{Row I} \\ \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This is the reduced echelon form

The general solution is
$$\begin{aligned} x_1 - 2x_2 \quad + x_4 &= 0 \\ x_3 - 2x_4 &= 0 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_2 - x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

the non-leading variables are the free variables

Every linear combination of $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ is an elt of Nul A, and vice versa.

$\boxed{\text{Thus } \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\} \text{ is a spanning set for Nul } A.}$ $\square$

Q: True or false? The set of all solutions to $\vec{A}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is a subspace of $\mathbb{R}^4$

Q: True or false? The set of all solutions to $\vec{A}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$ is a subspace of $\mathbb{R}^4$

Q: True or false? Let H be the set of all vectors in $\mathbb{R}^4$ whose coordinates a, b, c, d satisfy the equations $a - 2b + 5c = d$ and $c - a = b$. Then H is a subspace of $\mathbb{R}^4$.

Q: True or false? Let H be the set of all vectors in $\mathbb{R}^4$ whose coordinates a, b, c, d satisfy the equations $a - 2b + 5c = d + 1$ and $c - a = b$. Then H is a subspace of $\mathbb{R}^4$.

Part II : Column space

Def : The <u>column space</u> of an $m \times n$ matrix A is

$$\text{Col } A = \{ \text{all linear combinations of the columns of } A \},$$

i.e. $\text{Col } A = \text{Span}\{ \text{columns of } A \}$

i.e. $\text{Col } A = \{ \vec{b} : \vec{b} = \underbrace{A\vec{x}} \text{ for some } \vec{x} \text{ in } \mathbb{R}^n \}$

Notation $A\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ means the linear combination of the columns of A with weights $x_1, x_2, \ldots, x_n$

Thm 3 The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$

Pf Let $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ denote the columns of A.

Then $\text{Col } A = \text{Span}\{\vec{a}_1, \ldots, \vec{a}_n\}$, which is a subspace by Thm 1.

which says: "If $v_1, v_2, \ldots, v_p$ are in a vector space V, then $\text{Span}\{v_1, \ldots, v_p\}$ is a subspace of V"

Since the columns of A are in $\mathbb{R}^m$, the result follows. ☒

Ex: Let $W = \left\{ \begin{bmatrix} -y-z \\ y \\ z \end{bmatrix} : y, z \text{ are real numbers} \right\}$

Find a matrix A such that $W = \text{Col } A$.

Sol: 1. Write W as a set of linear combinations :

$$W = \left\{ y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} : y, z \text{ in } \mathbb{R} \right\} = \text{Span}\Bigg\{ \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{\text{A spanning set for } W} \Bigg\}$$

2. Use the vectors in the spanning set as the columns of A:

Let $A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Then $W = \text{Col } A$. □

Fact: Assume A is an $m \times n$ matrix. TFAE:

1. $A\vec{x} = \vec{b}$ has a solution for each $\vec{b}$ in $\mathbb{R}^m$ } Thm 4 in Sec 1.4
2. The columns of A span $\mathbb{R}^m$
3. $\text{Col } A = \mathbb{R}^m$

since $\text{Col } A = \text{Span}\{\text{columns of } A\}$

III. Row space

Def The *row space* of an $m \times n$ matrix A, denoted $\text{Row } A$ is the set of all linear combinations of the row vectors of A.

Thm $\text{Row } A$ is a subspace of $\mathbb{R}^n$

Note: Since rows of A = columns of A^T ← transpose

$\text{Row } A = \text{Col } A^T$

Ex: let $A = \begin{bmatrix} 1 & -2 & -1 & 3 \\ 2 & -4 & 1 & 0 \\ 1 & -2 & 2 & -3 \end{bmatrix}$ be as before.

1. Find a nonzero vector in $\text{Row } A$,
2. ——— " ——— $\text{Col } A$, and
3. ——— " ——— $\text{Nul } A$.
4. Determine wether $\vec{u} = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$ is in $\text{Col } A$

Sol: 1. Any linear combination of the row vectors of A, e.g

$[1 \ \ -2 \ \ -1 \ \ 3]$ first row

2. Any linear combination of the columns of A, e.g

$$\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \text{ or } \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} -1+6 \\ 1 \\ 2-6 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -4 \end{bmatrix}$$

3. Earlier, we computed that

$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$ is a spanning set for Nul A

So any linear combination of this set, e.g

$$\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad 2\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4+1 \\ 2 \\ 0+2 \\ 0-1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 2 \\ -1 \end{bmatrix}.$$

4. From Day 1 class notes, we have

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{array}\right] \xrightarrow[\text{reduce}]{\text{Row}} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} x_1 - 2x_2 \quad + x_4 &= 2 \\ x_3 - 2x_4 &= 1 \\ 0 &= 0 \end{aligned}$$

So $A\vec{x} = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$ has at least a solution,

so $\begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$ is in Col A.

IV. Kernel and range (image) of a linear map

Def Suppose V and W are vector spaces.

A map $T : V \longrightarrow W$ is called <u>linear</u> if

(i) T preserves addition, i.e.

$T(u+v) = T(u) + T(v)$ for all u, v in the domain V

(ii) T preserves scalar multiplication, i.e.

$T(cu) = c\,T(u)$ for all u in the domain V and all scalars c.

Def The **kernel** (or **null space**) of T is the set of all u in the domain V such that $T(u) = 0$

zero element in vector space W

Def The **range** (or **image**) of T is the set of all elements in the codomain W of the form $T(x)$ for some x in the domain V.

Thm
- The kernel of T is a subspace of V
- The range of T is a subspace of W

Note:
- $\ker T = \{$zero element$\}$ if and only if T is one-to-one
- range $T = W$ if and only if T is onto.

Note: If $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a matrix transformation defined by $T(\vec{x}) = A\vec{x}$, then:

The kernel of T is just the null space of A

The range of T is just the column space of A

Ex (Ex 9)
- Let $V = \{$Differentiable functions $f: [0,1] \to \mathbb{R}$ whose derivatives are continuous$\}$

  $W = \{$Continuous functions $f: [0,1] \to \mathbb{R}\}$
- Define $D: V \to W$ to be the map

  $$D(f) = f'.$$
- This is a linear map: $D(f+g) = D(f) + D(g)$ ✓
  $D(cf) = cD(f)$ ✓
- The zero element in the codomain W is the zero constant function.
- So the kernel of D is $\{$constant functions$\}$, so D is <u>not</u> one-to-one.
- The range of D is the entire W. So D <u>is</u> onto.

Group Quiz, pg 2

(Similar to Sec 4.2 #44 in MML)

Define a linear map $T: \mathbb{P}_2 \to \mathbb{R}^3$

by $T(p) = \begin{bmatrix} p(5) \\ p(5) \\ p(5) \end{bmatrix}$

a.) What is the kernel of T?

Find two polynomials in $\mathbb{P}_2$ that span the kernel of T.

b) What is the range of T?

Find a vector in $\mathbb{R}^2$ which spans the range of T

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