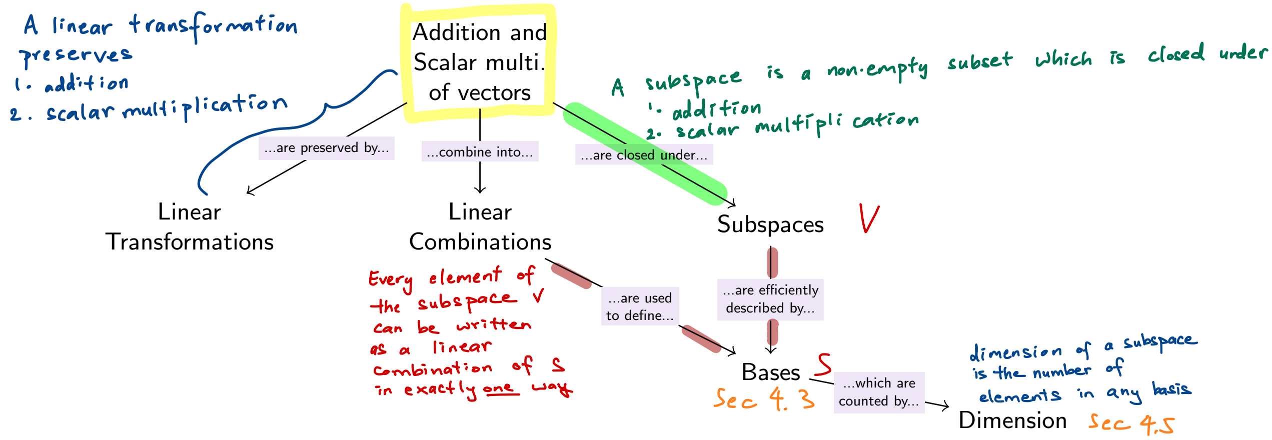
Sec 4.1 Vector spaces and subspaces

- Observation 1: Many linear algebra concepts can be defined in terms of addition and scalar multiplication



- Observation 2: Addition and scalar multiplication make sense for many other mathematical objects.

Examples

- Polynomials $\qquad \overbrace{(1+x^2)+(7-3x+x^3)}^{\text{polynomial addition}} \qquad \overbrace{4(1+3x+4x^2)}^{\text{scalar multiplication}}$
- Real-valued functions of x $\qquad \sin(x)+e^x \qquad \underbrace{4\ln(x)}_{\text{scalar multiplication}}$

function addition

4 is a scalar/number

Vector space

Goals: Generalize what we've learned so far about vectors to other kinds of objects we can add and scalar multiply.

Definition: A vector space

A **vector space** is a set V in which

- there is a rule to add any two elements v, w in V, and
- there is a rule to multiply any v in V by any scalar r in $\mathbb{R}$,

such that the axioms on the next slide hold.

Intuitively, a vector space is a set of mathematical objects which collectively behave like a set of vectors.

Possibly confusing terminology

- Elements of a vector space may not be vectors in $\mathbb{R}^n$
- Some textbooks (like ours) use 'vector' to refer to any element of a vector space.

Axioms for vector space

Axioms (essential properties) of addition

- $u + v = v + u$ for all u, v in V. (addition is commutative)
- $(u + v) + w = u + (v + w)$ for all u, v, w in V. (addition is associative)
- There is an element 0 in V, such that for all v in V, (additive identity, called "0", exists)
$$v + 0 = 0 + v = v$$
- For each v in V, there exists $-v$ in V with
$$v + (-v) = (-v) + v = 0$$
(additive inverse, denoted by "$-$", exists)

Axioms (essential properties) of scalar multiplication

- $r(u + v) = ru + rv$ for all u, v in V and any r in $\mathbb{R}$.
- $(r + s)v = rv + sv$ for all v in V and any r, s in $\mathbb{R}$.

distributivity

- $r(sv) = (rs)v$ for all v in V and any r, s in $\mathbb{R}$. (multiplication is associative)
- There is an element 1 such that $1v = v$ for all v in V. (multiplicative identity, called "1", exists)

An axiom is a fact that can't be reduced to a simpler property.

Two examples of vector spaces: $\mathbb{R}^n$ and $\mathbb{P}$

The set of vectors of height n is a vector space!

Fact (The motivating examples of vector spaces)

For each positive integer n, the set $\mathbb{R}^n$ is a vector space.

Fact (Our first non-vector vector space)

The set of polynomials in x is a vector space, denoted $\mathbb{P}$.

Useful fact: Two polynomials are equal if and only if they have the coefficients when written in standard form: $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$.

- **Exercise:** Determine whether $(x - 4)^3$ is a **scalar multiple** of $x^2 + x + 1$.

Is there a scalar c in $\mathbb{R}$ such that $(x-4)^3 = c(x^2+x+1)$?

First, put the LHS in standard form: $(x-4)(x^2-8x+16) = c(x^2+x+1)$

$x^3 - 12x^2 + (32+16)x + 64 = c(x^2+x+1)$

Since x^2+x+1 has no x^3 term, this is impossible.

$\therefore$ $(x-4)^3$ is not a scalar multiple of x^2+x+1.

Exercise

- Write x^2 as a **linear combination** of 1, $1+x$, and $1+2x+x^2$.

(Note: Numbers like 0, 1, and 7 count as **constant** polynomials!)

We want to find a, b, c in $\mathbb{R}$ such that

$$x^2 = a\cdot 1 + b(1+x) + c(1+2x+x^2).$$

Note: This is a polynomial in one variable, x. The letters a, b, c are just numbers we're trying to find.

Put the RHS into standard form, so that it's easy to compare the two sides.

$$x^2 = \underline{\quad\quad}\, x^2 + \underline{\quad\quad}\, x + \underline{\quad\quad}\, 1$$

Think of $1 = x^0$

We collect all terms with x^2, all terms with x, and all constant terms.

$$x^2 = \underbrace{c}\, x^2 + \underbrace{(b+2c)}\, x + \underbrace{(a+b+c)}\, 1$$

The only way the LHS equals RHS is if all coefficients match.

$$1\cdot x^2 + 0\, x + 0\cdot 1 = c\, x^2 + (b+2c)\, x + (a+b+c)\cdot 1$$

This tells us

$$\left.\begin{aligned} 1 &= c \\ 0 &= b + 2c \\ 0 &= a+b+c \end{aligned}\right\}$$

(a system of linear equations!)

$c = 1$

$b+2c=0 \Rightarrow b+2=0 \Rightarrow b=-2$

$a+b+c=0 \Rightarrow a+(-2)+1=0 \Rightarrow a-1=0 \Rightarrow a=1$

check: $1-2(1+x)+1+2x+x^2 \stackrel{?}{=} x^2$ ✓

$$\therefore\ \boxed{x^2 = \underbrace{1}_{a}\cdot(1) + \underbrace{(-2)}_{b}(1+x) + \underbrace{(1)}_{c}(1+2x+x^2)}$$

Definition: The degree of a polynomial

The **degree** of a non-zero polynomial in x is the largest power of x with non-zero coefficient.

We define $\deg(0) := -\infty$, mostly to avoid an extra case.

Fact (Polynomials of degree at most n)

For each positive integer n, the set of polynomials in x of degree at most n is a vector space, denoted $\mathbb{P}_n$.

Examples:

- $\mathbb{P}_1$ consists of polynomials $ax+b$, for a, b in $\mathbb{R}$.
- The three polynomials $(x-1)^3$, x^2+3x, and 2 are in $\mathbb{P}_3$, but the polynomials x^4 and x^8-2x^3 are not.
- $\mathbb{P}_0$ is just the constant polynomials like 0, 1, and 7, which are the same as numbers, so $\mathbb{P}_0 = \mathbb{R}$.

A non-example: Consider the set of polynomials of degree exactly 3.

Consider $S := \{$ polynomials of degree exactly 3 $\}$.
Then x^3+x and $-x^3$ are in S,
but their sum is not.
So S is not closed under addition.

By a **sequence**, we mean an infinite list of real numbers.

Examples of sequences

$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$ (the Fibonacci sequence)

$2, 3, 5, 9, 11, 13, 17, \ldots$ (prime numbers)

$1, 3, 9, 27, 81, 243, \ldots$ (powers of 3)

$7, 12, -5, \pi, 3.5, 7, \ldots$ (Just some random numbers)

Fact (The set of sequences is a vector space)

The set of sequences is a vector space, denote $\mathbb{S}$. Addition and scalar multiplication are defined term-wise.

Fact (Sets of matrices of fixed size are vector spaces)

For positive integers m and n, the set of $m \times n$-matrices is a vector space, denoted $\mathbb{R}^{m\times n}$.

Addition and scalar multiplication are the matrix versions.

Example: Let's say we know $\mathbb{P}$ is a vector space, but not $\mathbb{P}_3$.

- To add two polynomials in $\mathbb{P}_3$, add them as polynomials in $\mathbb{P}$, and observe the result is still in $\mathbb{P}_3$.
- Scalar multiplication also does not leave $\mathbb{P}_3$.

Since the axioms hold in $\mathbb{P}$, they automatically hold in $\mathbb{P}_3$. **So $\mathbb{P}_3$ is a vector space.**

Definition: Subspace of a vector space

Let V be a vector space. A **subspace** of V is a non-empty subset W of V which is...

a. Show the zero element is in W

b. • **closed under addition**; that is,

$$\text{for all } v, w \text{ in } W, \text{ the sum } v + w \text{ is in } W, \text{ and}$$

c • **closed under scalar multiplication**; that is,

$$\text{for all } v \text{ in } W \text{ and } c \text{ in } \mathbb{R}, \text{ the product } cv \text{ is in } W.$$

Fact (Subspaces are vector spaces)

A subspace of a vector space is also a vector space.

Exercise

Let S denote the set of polynomials in $\mathbb{P}_2$ such that $f(5) = 0$. That is,

$$S = \{f(x) \text{ in } \mathbb{P}_2 \mid f(5) = 0\}.$$

Show whether S is a subspace or not a subspace of $\mathbb{P}_2$.

SAMPLE STUDENT ANSWER

a. The polynomial $\underbrace{x-5}_{\text{degree is } 1 \leq 2}$ is in $\mathbb{P}_2$ and plugging in 5 into $x-5$ gives 0.
So $x-5$ is in S. This shows S is nonempty.

b. Let f and g be in S.
That is, f and g are polynomials with degree 2 or smaller, and

$f(5)=0$
$g(5)=0.$

So $f+g$ is a polynomial with degree 2 or smaller, and

$$(f+g)(5) = f(5)+g(5) = 0+0 = 0$$

Therefore $f+g$ is in S. So S is closed under addition.

c. Let c be in $\mathbb{R}$ and let f be in S.
That is, f is a polynomial in x with degree 2 or smaller, and
$f(5)=0.$

Then cf is also in $\mathbb{P}_2$ and

$$(cf)(5) = c.f(5) = c.0 = 0$$

So cf is in S. Therefore S is closed under scalar multiplication.

Thus, S is a subspace of $\mathbb{P}_2$.

— the end —

Exercise

Let T denote the set of polynomials in $\mathbb{P}_2$ such that $f(5) = 1$. That is,

$$T = \{f(x) \text{ in } \mathbb{P}_2 \mid f(5) = 1\}.$$

Show whether T is a subspace or not a subspace of $\mathbb{P}_2$.

SAMPLE STUDENT ANSWER

Let $f(x) := x-4$, which is in T.

Then $(f+f)(x) = x-4+x-4 = 2x-8$

so $(f+f)(5) = 2(5)-8 = 2$

Since $(f+f)(5) \neq 1$, $f+f$ is not in T.
So T is not closed under addition.

Therefore T is not a subspace of $\mathbb{P}_2$.

— the end —

ANOTHER SAMPLE STUDENT ANSWER

Let $f(x) := x-4$, which is in T.

Then $(4f)(x) = 4(x-4) = 4x-16.$

So $(4f)(5) = 20-16 = 4$

Since $(4f)(5) \neq 1$,
$4f$ is not in T.

So T is not closed under scalar multiplication.

Therefore T is not a subspace of $\mathbb{P}_2$.

— the end —

III. A subspace spanned by a set

Definition: Span$\{v_1, v_2, \ldots, v_p\}$

The **span** of a set of objects is the set of their linear combinations.

Example:

$$\text{Span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-10\\0\\-4\end{bmatrix}\right\} := \left\{t\begin{bmatrix}1\\1\\0\end{bmatrix} + s\begin{bmatrix}-10\\0\\-4\end{bmatrix} \text{ for all } t, s \text{ in } \mathbb{R}\right\}$$

$$\textbf{Span}\left\{\begin{bmatrix}1\\4\\7\end{bmatrix}, \begin{bmatrix}2\\5\\8\end{bmatrix}, \begin{bmatrix}3\\6\\9\end{bmatrix}\right\} := \left\{r\begin{bmatrix}1\\4\\7\end{bmatrix} + s\begin{bmatrix}2\\5\\8\end{bmatrix} + t\begin{bmatrix}3\\6\\9\end{bmatrix} \text{ for all } r, s, t \text{ in } \mathbb{R}\right\}$$

Theorem: Spans are subspaces

The span of a set of objects in a vector space V is a subspace of V.

- Span$\{v_1, v_2, \ldots, v_p\}$ is called **the subspace** spanned by $\{v_1, v_2, \ldots, v_p\}$.

Definition: Spanning sets

A **spanning set** (or **generating set**) of H is a set of objects whose span is H.

- If $H = \text{Span}\{v_1, v_2, \ldots, v_p\}$, then $\{v_1, v_2, \ldots, v_p\}$ is a **spanning set** for H

Exercise

Let W be the subset of $\mathbb{R}^3$ consisting of vectors whose second entry is the average of the other two. Show that W is a subspace of $\mathbb{R}^3$.

Sol: W is the set of all vectors of the form

$$\begin{bmatrix}a\\\frac{a+b}{2}\\b\end{bmatrix} = \begin{bmatrix}a\\\frac{1}{2}a+\frac{1}{2}b\\b\end{bmatrix} = a\begin{bmatrix}1\\\frac{1}{2}\\0\end{bmatrix} + b\begin{bmatrix}0\\\frac{1}{2}\\1\end{bmatrix}$$

where a, b are real numbers.

This implies that $W = \text{Span}\left\{\begin{bmatrix}1\\\frac{1}{2}\\0\end{bmatrix}, \begin{bmatrix}0\\1/2\\1\end{bmatrix}\right\}$,

so W is a subspace of $\mathbb{R}^3$.