2.2 The inverse of a matrix

$$I_n \overset{\text{def}}{=} \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{bmatrix}$$

Def If A is an $n \times n$ matrix, the <u>inverse</u> of A is the $n \times n$ matrix C such that:

- $AC = I_n$
- $CA = I_n$

The inverse of A is denoted by A^{-1}

Def If A has an inverse, A is called <u>invertible</u> or <u>non-singular</u>.

If A^{-1} doesn't exist, A is called <u>not invertible</u> or <u>singular</u>.

Note: Non-square matrices are never invertible.

Ex: $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$, $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$.

$$AC = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2\cdot -7 + 5\cdot 3 & 2\cdot -5 + 5\cdot 2 \\ -3\cdot -7 - 7\cdot 3 & -3\cdot -5 - 7\cdot 2 \end{bmatrix} = \begin{bmatrix} -14+15 & -10+10 \\ 21-21 & 15-14 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$CA = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So $C = A^{-1}$ and $A = C^{-1}$

An algorithm for finding A^{-1}

- Row reduce the augmented matrix $[A \mid I]$
- If A is row equivalent to I, then $[A \mid I] \underset{\text{Row}}{\sim} [I \mid A^{-1}]$

  If not, A doesn't have an inverse

Ex: Find the inverse of $A = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$, if it exists.

$$[A \mid I] = \left[\begin{array}{cc|cc} -1 & 2 & 1 & 0 \\ 3 & -6 & 0 & 1 \end{array}\right] \xrightarrow[3R_1 + R_2]{} \left[\begin{array}{cc|cc} -1 & 2 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{array}\right]$$

Note: $\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}$ is a (row) echelon form that is row equivalent to A

There is only one pivot, so it is not row equivalent to $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

which has two pivots. So A is not row equivalent to I.

Thus A is <u>not invertible</u>

Ex (Sec 2.2 Example 7 pg 117)

Find the inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, if it exists

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right]$$

This is a row echelon form that is row equivalent to A. Since it has 3 pivots, it's row equivalent to I_3. So $A \underset{\text{Row}}{\sim} I_3$, so we can keep going.

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right] = [I \mid A^{-1}]$$

A^{-1} //

Thm (Sec 2.2 Thm 6)

Let A and B be $n \times n$ matrices. Suppose A, B are invertible.

i) Then A^{-1} is also invertible, and $(A^{-1})^{-1} = A$

ii) ("Socks-shoes" property)

Then AB is also invertible, and $(AB)^{-1} = B^{-1}A^{-1}$

iii) Then A^T is also invertible, and $(A^T)^{-1} = (A^{-1})^T$

(the transpose of A)

The inverse of A^T is the transpose of A^{-1}

Proof of (ii): $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$ since matrix multiplication is associative

$= A\,I\,A^{-1}$ since B^{-1} is the inverse of B

$= AA^{-1}$

$= I$

We can also compute $(B^{-1}A^{-1})(AB) = I$

This shows that $B^{-1}A^{-1}$ is the inverse of AB $\boxtimes$

$\boxed{\text{Sec 2.3 Characterizations of Invertible matrices}}$

I.

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.

b. A is row equivalent to the $n \times n$ identity matrix.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.

<u>Thm</u> (The invertible matrix theorem)

Let A be an $n \times n$ matrix.

The following are equivalent

a. A has an inverse

b. A is row equivalent to I_n

c. A has n pivot positions

d. The homogeneous equation $A\vec{x} = \vec{0}$ has only the trivial solution.

Proof of (c) $\Rightarrow$ (d) (Read "(c) implies (d)" or "If (c) holds then (d) holds")

Since A has n pivot positions, the augmented matrix

$$\left[\begin{array}{c|c} A & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right] \text{ is row equivalent to } \left[\begin{array}{cccc|c} 1 & & & 0 & 0 \\ 0 & 1 & & & 0 \\ 0 & 0 & 1 & & \vdots \\ & 0 & & \ddots & 0 \\ & & & 1 & 0 \end{array}\right]$$

so $A\vec{x} = \vec{0}$ has exactly one solution $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

e. The columns of A are linearly independent

Proof of (4) $\Leftrightarrow$ (5) (Read "(4) is true if and only if (5) is true")

This follows from the definition of linear independence (in Sec 1.7)

f. The linear map $\vec{x} \mapsto A\vec{x}$ is one-to-one

g. For every $\vec{b}$ in $\mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has at least one solution.

h. The columns of A span $\mathbb{R}^n$

i. The linear map $\vec{x} \mapsto A\vec{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$

l. A^T is invertible.

Ex: Determine if $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$ is invertible. (Extra)

Ans: A is row equivalent to $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

So it has 1 pivot position.

To be invertible, a 4×4 matrix must have 4 pivots.

So A is not invertible.

(Sec 2.3 Exercise # 21 & 22) in MML

Ex A <u>upper triangular matrix</u> is a matrix whose entries below the main diagonal are 0's, e.g.

$$\begin{bmatrix} 1 & 0 & 3 & 6 \\ 0 & 5 & 7 & 8 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 3 & 5 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

When is a square upper triangular matrix invertible?

Sol: By the Invertible Matrix Theorem,

an $n \times n$ matrix A is invertible if and only if A has n pivot positions.

If all entries in the main diagonal are nonzero, then A is already in row echelon form, and we see A has n pivot positions.

If one of the entries in the main diagonal is 0, say, in column k, then column k cannot be a pivot column, so A has fewer than n pivot positions.

Ans An $n \times n$ upper triangular matrix A is invertible if and only if all entries on the main diagonal of A are nonzero.

II. Invertible linear maps

Def If $T: \mathbb{R}^n \to \mathbb{R}^n$ is a linear map, the <u>inverse</u> of T is a function $S: \mathbb{R}^n \to \mathbb{R}^n$ such that

(1) $S(T(\vec{x})) = \vec{x}$ for all $\vec{x}$ in $\mathbb{R}^n$

(2) $T(S(\vec{x})) = \vec{x}$ for all $\vec{x}$ in $\mathbb{R}^n$

The inverse of T is denoted by T^{-1}.

Def A linear map $T: \mathbb{R}^n \to \mathbb{R}^n$ is called <u>invertible</u> if T has an inverse map.

Note: A linear map $T: \mathbb{R}^n \to \mathbb{R}^m$ is never invertible if $n \neq m$.

Thm (Sec 2.3 Thm 9)

Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear map & let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix.

If T is invertible, the inverse of T is given by $S(\vec{x}) = A^{-1}\vec{x}$.

Ex (Sec 2.3 Exercise # 41 & 42) in MML

Consider the map $T : \mathbb{R}^2 \to \mathbb{R}^2$

defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -5x + 9y \\ 4x - 7y \end{bmatrix}$

- Is T linear? Ans: yes, since $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -5 & 9 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

  so T is defined by matrix multiplication.

- Is T invertible? If so, find a formula for T^{-1}.

Ans: The standard matrix for T is $\begin{bmatrix} -5 & 9 \\ 4 & -7 \end{bmatrix}$.

Use the algorithm $[A \mid I] \to$ Row reduce:

$$[A \mid I] = \left[\begin{array}{cc|cc} -5 & 9 & 1 & 0 \\ 4 & -7 & 0 & 1 \end{array}\right] \xrightarrow[\frac{4}{5}R_1 + R_2]{} \left[\begin{array}{cc|cc} -5 & 9 & 1 & 0 \\ 0 & \frac{1}{5} & \frac{4}{5} & 1 \end{array}\right] \xrightarrow[5R_2]{} \left[\begin{array}{cc|cc} -5 & 9 & 1 & 0 \\ 0 & 1 & 4 & 5 \end{array}\right]$$

We see A is invertible, so we keep going

$$\xrightarrow{-9R_2 + R_1} \left[\begin{array}{cc|cc} -5 & 0 & -35 & -45 \\ 0 & 1 & 4 & 5 \end{array}\right] \xrightarrow{-\frac{1}{5}R_1} \left[\begin{array}{cc|cc} 1 & 0 & 7 & 9 \\ 0 & 1 & 4 & 5 \end{array}\right] = [I \mid A^{-1}]$$

So T is invertible. Its inverse is the linear map

$T^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 7x + 9y \\ 4x + 5y \end{bmatrix}. \quad \boxtimes$$

(Extra)

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\det A = ad - bc$

Thm 4 (Sec 2.2)

If $\det(A) \neq 0$ then A is invertible with

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $\det(A) = 0$ then A is not invertible.