Sec 1.9 The matrix of a linear transformation

Big Thm: Every linear map from $\mathbb{R}^n \to \mathbb{R}^m$ can be described by a matrix transformation $\vec{x} \mapsto A\vec{x}$

I. Standard matrix for the linear transformation

- To find this matrix A, it's enough to look at the columns of the $n \times n$ identity matrix I_n

$$\vec{e}_1 = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\text{height } n}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \dots, \ \vec{e}_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{k-th position}, \ \dots, \ \vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}$$

Ex:

- Suppose we know T is a linear transformation $\mathbb{R}^3 \to \mathbb{R}^2$ such that $T(\vec{e}_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $T(\vec{e}_3) = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.
- With no additional information, find a formula for the image $T(\vec{x})$ for an arbitrary $\vec{x}$ in $\mathbb{R}^3$.

Sol: Write $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + x_3 \vec{e}_3$

Then we have

$$T(\vec{x}) = T(x_1 \vec{e}_1 + x_2 \vec{e}_2 + x_3 \vec{e}_3)$$

$$= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + x_3 T(\vec{e}_3) \quad \text{since } T \text{ is linear}$$

$$= x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + 3x_2 + 5x_3 \\ 2x_2 + 4x_2 + 6x_3 \end{bmatrix}$$

Thm (Thm 10)

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map.

Then there is a $\underbrace{\text{unique}}_{\text{meaning, exactly one}}$ $m \times n$ matrix A such that

$$T(\vec{x}) = A\vec{x} \quad \text{for all } \vec{x} \text{ in } \mathbb{R}^n.$$

This unique matrix is $A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix}$ is

called the standard matrix for T

the concatenation of the vectors $T(\vec{e}_1), \dots, T(\vec{e}_n)$.

Ex

- Suppose we know T is a linear transformation $\mathbb{R}^3 \to \mathbb{R}^2$ such that $T(\vec{e}_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $T(\vec{e}_3) = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.
- Find the standard matrix for T

Sol: $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

II. Geometric linear maps of $\mathbb{R}^2$

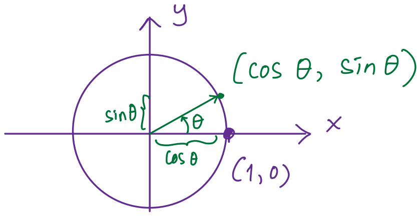
Pg 2

Rotation

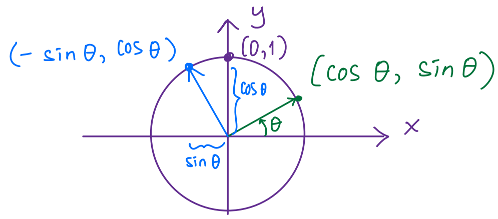
The map $R: \mathbb{R}^2 \to \mathbb{R}^2$ that rotates each point in $\mathbb{R}^2$ about the origin through an angle θ (counterclockwise for positive θ) is a linear map.

What is the standard matrix A of this rotation R?

Sol:



means $R\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$



means $R\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$

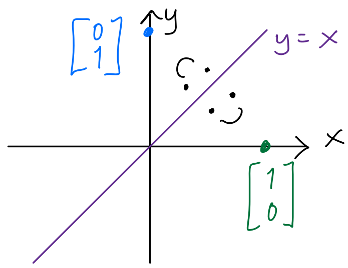
So $A = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

E.g. if $\theta = \frac{\pi}{2}$ then $A = \begin{bmatrix}\cos\frac{\pi}{2} & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$

Reflection

The map $F: \mathbb{R}^2 \to \mathbb{R}^2$ that reflects each point in $\mathbb{R}^2$ through a line ℓ (that passes through the origin) is a linear map.

Ex: The standard matrix for the reflection through the line $y = x$ is ... $A = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$



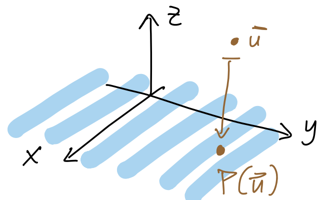
F maps $\begin{bmatrix}1\\0\end{bmatrix}$ to $\begin{bmatrix}0\\1\end{bmatrix}$
$\vec{e}_1$

and $\vec{e}_2$ to $\begin{bmatrix}1\\0\end{bmatrix}$

Projection

The mapping $P: \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} \longmapsto \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x\\y\\0\end{bmatrix}$$

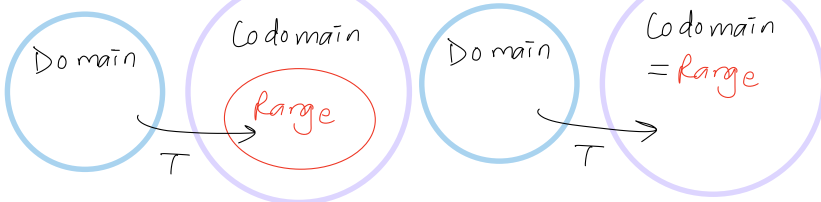


is called a projection because P projects points in $\mathbb{R}^3$ onto xy-plane

III. Existence & uniqueness questions: surjectivity

Def A map $T : \mathbb{R}^n \to \mathbb{R}^m$ is called onto $\mathbb{R}^m$ if...
for every $\vec{b}$ in $\mathbb{R}^m$, there is at least one $\vec{x}$ in $\mathbb{R}^n$

Cartoon



T is not onto

T is onto

Ex The projection $P : \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

is not onto. For example, $\vec{b} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$ is in codomain $\mathbb{R}^3$

but $\vec{b}$ is not the image $P(\vec{x})$ of any $\vec{x}$ in domain $\mathbb{R}^3$.

Ex However, the projection $P' : \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is onto. For any $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in $\mathbb{R}^2$,

we have $P'\left(\begin{bmatrix} x_1 \\ x_2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

so every $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in $\mathbb{R}^2$ is in the image of P'.

In general, we can answer this existence question by looking at an echelon form. pg 4

Thm

Let A be an $m \times n$ matrix.

Then the following are equivalent (TFAE):

I. For each $\vec{b}$ in $\mathbb{R}^m$, the equation $A\vec{x}=\vec{b}$ has a solution

II. The columns of A span $\mathbb{R}^m$

III. A has a pivot in every row

Thm 4 Sec 1.4 pg 39

IV (New in Sec 1.9) The map
$$\mathbb{R}^n \to \mathbb{R}^m$$
$$\vec{x} \mapsto A\vec{x}$$
maps onto $\mathbb{R}^m$

<u>Ex</u> If T is the linear transformation whose standard matrix can be row-reduced to

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 0 & 8 \end{bmatrix} \quad \text{then } T \text{ is } \underline{\underline{\text{onto}}}$$

Reason: Every row has a pivot.

<u>Ex</u> If T is the linear transformation whose standard matrix can be row-reduced to

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{then } T \text{ is } \underline{\underline{\text{not}}} \text{ onto}$$

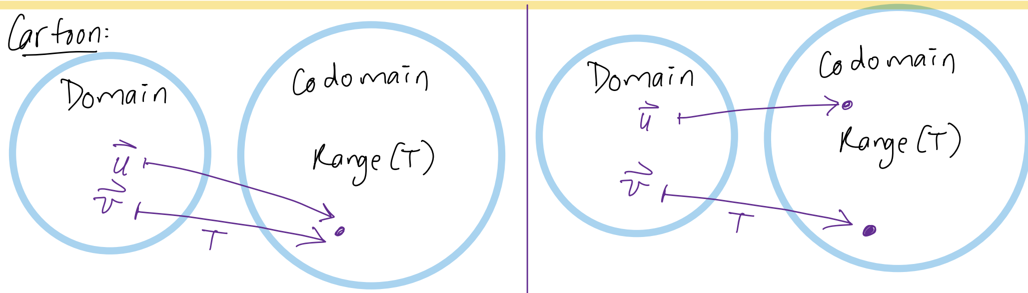
Reason: Row 3 has no pivot.

IV. Existence & uniqueness questions: injectivity

Def A map $T: \mathbb{R}^n \to \mathbb{R}^m$ is called one-to-one if..

if $T(\vec{u}) = T(\vec{v})$ then $\vec{u} = \vec{v}$.

Cartoon:



Not one-to-one   one-to-one

Ex The projection $P: \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

is not 1-1. For example, $P\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = P\left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right)$

Thm (Thm 11) Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.
T is one-to-one if and only if
$T(\vec{x}) = \vec{0}$ has only the trivial solution.

Ex The projection $P': \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is also not 1-1. Reason: $P'\left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

So $P'(\vec{x}) = \vec{0}$ has solutions other than the trivial solution.

Thm (eq (3) in Sec 1.7, pg 61)

TFAE:

1. $A\vec{x} = \vec{0}$ has only the trivial solution
2. The columns of A are linearly independent
3. # of pivots of A = # of columns of A
4. (New Sec 1.9) The map $\vec{x} \mapsto A\vec{x}$ is one-to-one

<u>Ex</u> If T is the linear transformation whose standard matrix can be row-reduced to

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

then T is <u>not</u> 1-1

Reason: The augmented matrix for $A\vec{x} = \vec{0}$ can be row-reduced to

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 5 & 6 & 7 & 0 \\ 0 & 0 & 0 & 8 & 0 \end{array}\right]$$

there is one free variable for the solution set, so $A\vec{x} = \vec{0}$ has more than just the trivial sol.

Ex If T is the linear transformation whose standard matrix can be row-reduced to

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ then T is also not 1-1.

non-pivot columns

Reason: The solution set of $A\vec{x} = \vec{0}$ has two free variables, so the columns of A are not linearly independent

Ex If T is the linear transformation whose standard matrix is $A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 0 & 0 \end{bmatrix}$

then T is 1-1.

Reason: By inspection, we see the columns of A are linearly independent because $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ is not a scalar multiple of $\begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$ and vice versa.

end