Sec 1.8 Intro to linear transformations

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I. Matrix transformation

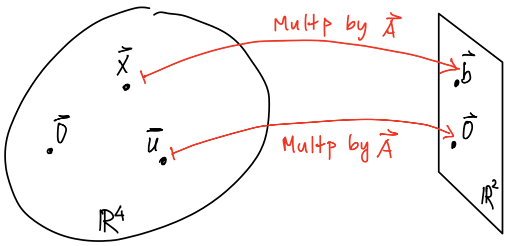
Think of a matrix A as "acting on" a vector $\vec{x}$ by multiplication to produce a new vector $A\vec{x}$.

Ex: $A = \begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix}$ Note: $\vec{x}, \vec{u}$ are in $\mathbb{R}^4$

$$A\vec{x} = \begin{bmatrix} 4-3+1+3 \\ 2 \quad +5+1 \end{bmatrix} = \underbrace{\begin{bmatrix} 5 \\ 8 \end{bmatrix}}_{\text{Call this } \vec{b}}$$

$$A\vec{u} = \begin{bmatrix} 4(1)-3(4)+1(-1)+3(3) \\ 2(1) \quad +5(-1)+1(3) \end{bmatrix} = \begin{bmatrix} 4-12-1+9 \\ 2 \quad -5+3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\text{zero vector } \vec{0}}$$

Here, multiplication by A transforms $\vec{x}$ into $\vec{b}$ and transforms $\vec{u}$ into $\vec{0}$.



This transformation $\vec{x} \mapsto A\vec{x}$ is a function from a set of vectors to another set of vectors.

Def 1 • A transformation (or function or mapping)

$$T : \underbrace{\mathbb{R}^n}_{\text{domain}} \longrightarrow \underbrace{\mathbb{R}^m}_{\text{codomain}}$$

is a rule that assigns to each vector $\vec{x}$ in $\mathbb{R}^n$ a vector $\underbrace{T(\vec{x})}_{\text{the image of } \vec{x} \text{ (under the action } T)}$ in $\mathbb{R}^m$.

• The image of T (or the range of T) is the set of all images $T(\vec{x})$.

<u>Def</u> 2 A <u>matrix transformation</u> is a function

$$T : \underbrace{\mathbb{R}^n}_{\text{domain}} \longrightarrow \underbrace{\mathbb{R}^m}_{\text{codomain}}$$

$$T(\vec{x}) = A\vec{x}$$

where A is an $m \times n$ matrix (n columns, m rows)

Note: Each image $T(\vec{x})$ is of the form $A\vec{x}$,

i.e. $T(\vec{x})$ is a linear combination of the columns of A,

i.e. $T(\vec{x})$ is in the span of the columns of A.

Ex 1: Define a mapping $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ by

2 cols 3 rows

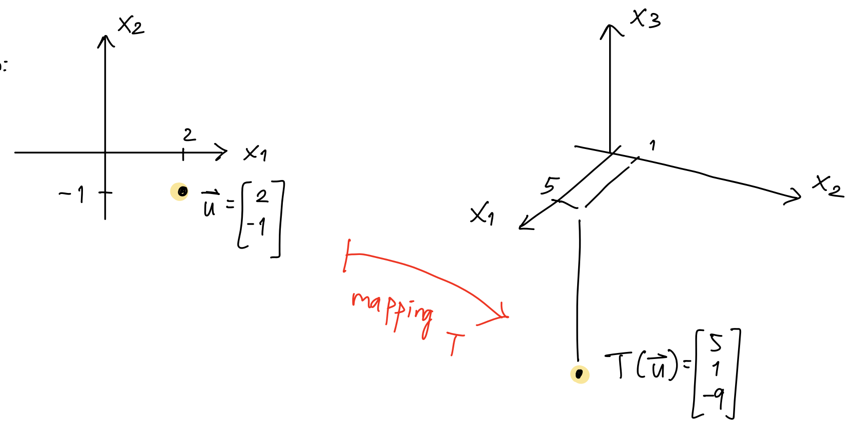
$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix}$$

↳ call this 3x2 matrix A

a. Let $\vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Find the image $T(\vec{u})$ of $\vec{u}$ under the mapping T.

Sol: $T(\vec{u}) = 2\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} - 1\begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix} + \begin{bmatrix} 3 \\ -5 \\ -7 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$

Sketch:



b. Let $\vec{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$ (i) Is $\vec{b}$ in the range of T ?

(ii) How many $\vec{x}$ are there in the domain $\mathbb{R}^2$ whose image under T is $\vec{b}$?

Sol: Write the augmented matrix associated to $A\vec{x} = \vec{b}$.

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{array}\right] \xrightarrow{\text{Row reduce}} \cdots \rightarrow \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{array}\right] \quad \left.\begin{array}{r} x_1 - 3x_2 = 3 \\ 2x_2 = -1 \end{array}\right\}$$

A $\vec{b}$ (echelon form)

- There is no row of the form [0 0 | nonzero] so $A\vec{x} = \vec{b}$ has at least one solution. (i) So $\vec{b}$ is in the range of T.
- Every column is a pivot column, so there are no free variables, meaning (ii) there is exactly one $\vec{x}$ in $\mathbb{R}^2$ whose image $T(\vec{x})$ is $\vec{b}$.

c. Let $\vec{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$. Is $\vec{c}$ in the range of T ? (Extra Ex)

Sol: Write the augmented matrix associated to $A\vec{x} = \vec{c}$.

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{array}\right] \xrightarrow{\text{Row reduce}} \cdots \rightarrow \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{array}\right] \quad \begin{array}{r} x_1 - 3x_2 = 3 \\ x_2 = 2 \\ 0 = -35 \end{array}$$

A (echelon form)

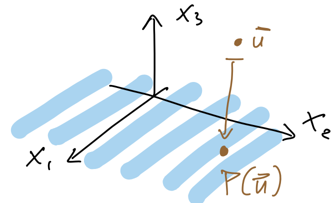
- The last row [0 0 | -35] tells us $A\vec{x} = \vec{c}$ is inconsistent. So $\vec{c}$ is NOT in the range of T.

Ex 2: (Extra Ex)

The mapping $P: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

is called a projection because P projects points in $\mathbb{R}^3$ onto $x_1 x_2$-plane



Ex 3: (Extra Ex)

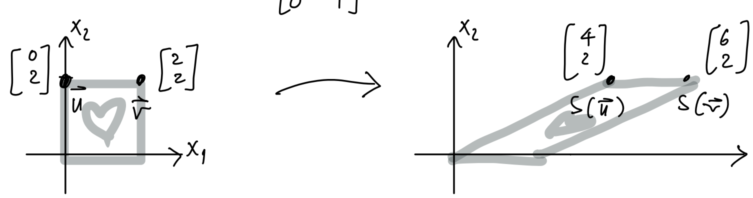
peh·ruh·leh·luh·gram

A shear mapping transform a square into a parallelogram

ex $S: \mathbb{R}^2 \to \mathbb{R}^2$

$$\vec{x} \longmapsto \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \vec{x}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2+4 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$



Top of the square is pushed to the right while the base is fixed

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II. Linear transformations

Ex of linear operations:
- Matrix transformations
- Derivatives
- Antiderivatives
- Laplace transform

Ex of non-linear operation: Quadratic function

The most important class of mappings

Def Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a function.
domain codomain

(i) T preserves vector addition (or "T preserves addition")

if :

$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}$ in the domain of T

(ii) T preserves scalar multiplication

if :

$T(c\vec{u}) = c\,T(\vec{u})$ for all $\underbrace{\text{scalars}}_{\text{numbers}}$ c and all $\vec{u}$ in the domain of T.

The mapping T is called linear if

T preserves addition and scalar multiplication.

Some properties of linear mappings

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then

1. T sends the zero vector (in $\mathbb{R}^n$) to the zero vector (in $\mathbb{R}^m$)

Why? $T(\vec{0}) = T(0\,\vec{u}) \underset{\uparrow}{=} 0\,T(\vec{u}) = \vec{0}$

by property (ii)

2. T sends linear combinations to linear combinations

$T(c\vec{u} + d\vec{v}) = c\,T(\vec{u}) + d\,T(\vec{v})$

Why? $T(c\vec{u} + d\vec{v}) \underset{\uparrow}{=} T(c\vec{u}) + T(d\vec{v}) \underset{\uparrow}{=} c\,T(\vec{u}) + d\,T(\vec{v})$

by property (i) by property (ii)

Ex: Show that $T: \mathbb{R}^2 \to \mathbb{R}^3$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ x + 4 \\ 5y \end{bmatrix}$$

is linear or not linear

Exercise # 41

Sol 1: I see that $T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$

So it doesn't preserve scalar multiplication

Sol 2: Try $\vec{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$$T(\vec{u} + \vec{v}) = T\left(\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -9 \\ 4 \\ 15 \end{bmatrix}$$

$$T(\vec{u}) + T(\vec{v}) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 4 \\ 5 \end{bmatrix} + \begin{bmatrix} -6 \\ 4 \\ 10 \end{bmatrix} = \begin{bmatrix} -9 \\ 8 \\ 15 \end{bmatrix}$$

So $T(\vec{u} + \vec{v}) \neq T(\vec{u}) + T(\vec{v})$

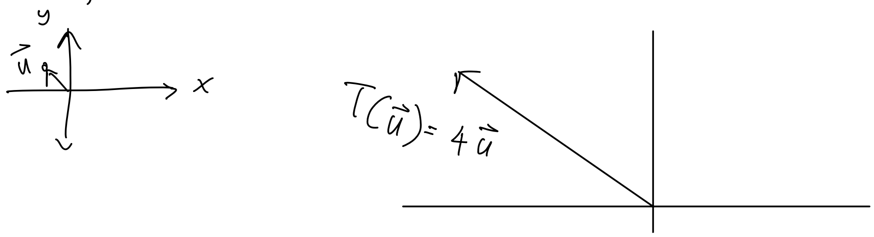
So T doesn't preserve vector addition.

Ex 4: Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 4\begin{bmatrix} x \\ y \end{bmatrix}$$

(Extra Ex)

(This map is called a <u>dilation</u>)



Show that T is a linear map.

Sol: Let $\vec{u}, \vec{v}$ be in $\mathbb{R}^2$ (domain of T)

Let c be a scalar

① Show that T preserves vector addition:

$$T(\vec{u}+\vec{v}) = 4(\vec{u}+\vec{v})$$

$$= 4\vec{u} + 4\vec{v}$$ by property of scalar multiplication

$$= T(\vec{u}) + T(\vec{v})$$

(ii) Show that T preserves scalar multiplication:

$$T(c\vec{u}) = 4(c\vec{u})$$

$$= 4c(\vec{u})$$

$$= c(4\vec{u})$$

$$= c\,T(\vec{u}).$$

Fact: ① $T : \mathbb{R}^n \to \mathbb{R}^m$
$\vec{x} \mapsto M\vec{x}$ is a linear map.

② So, if you can describe T as matrix multiplication, then T is automatically linear

— end of notes —