Math 2220 Notes for Ch 1 Review Handout

Consider the system of linear equations

$$\left.\begin{array}{rrrrl} x_1 & -2x_2 & -x_3 & +3x_4 & = 1 \\ 2x_1 & -4x_2 & +x_3 & & = 5 \\ x_1 & -2x_2 & +2x_3 & -3x_4 & = 4 \end{array}\right\} \quad (*)$$

① Sec 1.3 Vector equations

Write a vector equation that is equivalent to the system (*) (Sec 1.3)

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -4 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$$

② Sec 1.4 The matrix equation $A\vec{x} = \vec{b}$

Write a matrix equation that is equivalent to the system (*) (Sec 1.4)

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 2 & -4 & 1 & 0 \\ 1 & -2 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} \quad \text{or} \quad A\vec{x} = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$$

③ Sec 1.1 Systems of linear equations &

Sec 1.2 Row reduction & echelon forms

④ Is the system

$$\left.\begin{array}{rrrrl} x_1 & -2x_2 & -x_3 & +3x_4 & = 1 \\ 2x_1 & -4x_2 & +x_3 & & = 5 \\ x_1 & -2x_2 & +2x_3 & -3x_4 & = 4 \end{array}\right\} \quad (*)$$

consistent or inconsistent?

Sol:

Fact: A system of linear equations has

1. no solutions, or — the system is inconsistent
2. exactly one solution, or
3. infinitely many solutions

} the system is consistent

The augmented matrix of the system (*) is

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{array}\right] \quad (**)$$

Row reduce:

$$\begin{array}{l} -2\,\text{Row I} + \text{Row II} \\ -\text{Row I} + \text{Row III} \end{array} \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -6 & 3 \end{array}\right]$$

$$-\text{Row II} + \text{Row III} \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

called pivot positions

This matrix is in a (row) echelon form

Columns 1, 3 are called pivot columns

An echelon form of the augmented matrix gives us enough information to conclude that the system is consistent

(Thm 2 in Sec 1.2)
A linear system is consistent iff an echelon form of the augmented matrix has no row of the form $[0\ 0\ \cdots\ 0 \mid b]$ with b nonzero

Since an echelon form of the augmented matrix has no such row, the system (*) is consistent.

meaning there exist x_1, x_2, x_3, x_4 satisfying (*)

Optional: Continue to row reduce to obtain the unique reduced echelon form:

- leading entry in each nonzero row is 1
- Each leading 1 is the only nonzero entry in its column

$$\tfrac{1}{3}\,\text{Row II} \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\text{Row II} + \text{Row I} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \checkmark$$

5 Sec 1.3 & 1.4

Consider the matrix $A = \begin{bmatrix} 1 & -2 & -1 & 3 \\ 2 & -4 & 1 & 0 \\ 1 & -2 & 2 & -3 \end{bmatrix}$

Is $\begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$ in the subset of $\mathbb{R}^3$ spanned by the columns of A? Why/why not?

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -2 \\ -4 \\ -2 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ $\vec{v}_4 = \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}$

The subset of $\mathbb{R}^3$ spanned (or generated) by the columns of A denoted $\text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ is the collection of vectors that can be written in the form

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 + c_4 \vec{v}_4$$

} called linear combinations of $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$

with c_1, c_2, c_3, c_4 scalars

Since system (*) is consistent, there exist c_1, c_2, c_3, c_4 such that $c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 + c_4 \vec{v}_4 = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$

So $\begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$.

Thus $\begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$ is in Span (columns of A). (Answer to Q: yes!)

⑥ Sec 1.5 Solution sets of linear systems

ⓐ Describe all solutions of the nonhomogeneous system

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 2 & -4 & 1 & 0 \\ 1 & -2 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$$

$A\vec{x} = \vec{b}$ is called a nonhomogeneous system if $\vec{b}$ is not the zero vector

Sol:

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{array}\right] \xrightarrow{\text{Row reduce}} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

pivot columns

$$\begin{aligned} x_1 - 2x_2 \quad\quad + x_4 &= 2 \\ x_3 - 2x_4 &= 1 \\ 0 &= 0 \end{aligned}$$

Basic variables

The variables that don't correspond to pivot columns, x_2, x_4 are called free variables.

$$\boxed{\begin{cases} x_1 = 2x_2 - x_4 + 2 \\ x_2 \text{ is free} \\ x_3 = 2x_4 + 1 \\ x_4 \text{ is free} \end{cases} \quad \text{for all choices for } x_2, x_4}$$

or

$$\boxed{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{for all choices for } x_2, x_4}$$

ⓑ Describe all solutions of the homogeneous system

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 2 & -4 & 1 & 0 \\ 1 & -2 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

zero vector

$A\vec{x} = \vec{0}$ is called a homogeneous system

Sol:

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ 2 & -4 & 1 & 0 & 0 \\ 1 & -2 & 2 & -3 & 0 \end{array}\right] \xrightarrow{\text{Row reduce}} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_2 \quad\quad + x_4 &= 0 \\ x_3 - 2x_4 &= 0 \\ 0 &= 0 \end{aligned}$$

$$\boxed{\begin{cases} x_1 = 2s - t \\ x_2 = s \\ x_3 = 2t \\ x_4 = t \end{cases} \quad \text{for all choices for } s, t}$$

or

$$\boxed{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \quad \text{for all choices for } s, t}$$

(c) Does $A\vec{x} = \vec{0}$ have a nontrivial solution?

↳ not the zero vector

Yes, e.g., set $x_2 = 3$ and $x_4 = 0$ $\vec{x} = 3\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \\ 0 \end{bmatrix}$

is a nontrivial solution

Fact: A homogeneous linear system has at least one solution, $\vec{x} = \vec{0}$ called the trivial solution

A nontrivial solution is a nonzero vector $\vec{x}$ that satisfies $A\vec{x} = \vec{0}$

(7) Sec 1.7 Linear independence

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -2 \\ -4 \\ -2 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$, $\vec{v}_4 = \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}$

(a) Determine whether the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent.

Fact

The columns of a matrix A are linearly independent iff the matrix equation $A\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Since $A\vec{x} = \vec{0}$ has nontrivial solutions, the columns of A are not linearly independent (they are dependent)

(b) Determine by inspection whether $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent.

- I notice that $-2\vec{v}_1 = \vec{v}_2$, so
  $2\vec{v}_1 + \vec{v}_2 + 0\vec{v}_3 + 0\vec{v}_4 = \vec{0}$

- This means that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent by definition.

——— end of handout ———