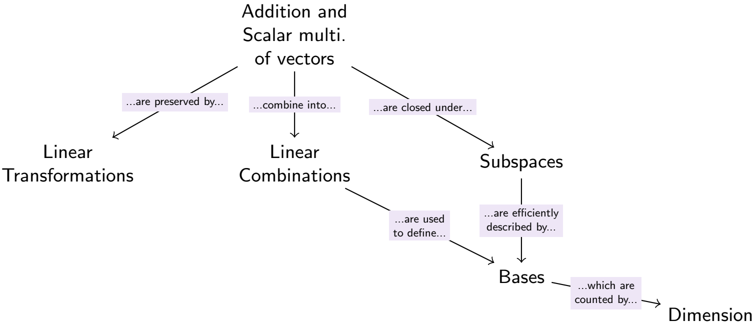
Sec 4.1 Vector spaces and subspaces

- Observation 1: Many linear algebra concepts can be defined in terms of addition and scalar multiplication



- Observation 2: Addition and scalar multiplication make sense for many other mathematical objects.

Examples

- Polynomials $(1 + x^2) + (7 - 3x + x^3)$ $4(1 + 3x + 4x^2)$
- Real-valued functions of x $\sin(x) + e^x$ $4\ln(x)$

Vector space

Goals: Generalize what we've learned so far about vectors to other kinds of objects we can add and scalar multiply.

Definition: A vector space

A **vector space** is a set V in which

- there is a rule to add any two elements v, w in V, and
- there is a rule to multiply any v in V by any scalar r in $\mathbb{R}$,

such that the axioms on the next slide hold.

Intuitively, a vector space is a set of mathematical objects which collectively behave like a set of vectors.

Possibly confusing terminology

- Elements of a vector space may not be vectors in $\mathbb{R}^n$
- Some textbooks (like ours) use 'vector' to refer to any element of a vector space.

Axioms for vector space

Axioms (essential properties) of addition

- $u + v = v + u$ for all u, v in V.
- $(u + v) + w = u + (v + w)$ for all u, v, w in V.
- There is an element 0 in V, such that for all v in V,
  $$v + 0 = 0 + v = v$$
- For each v in V, there exists $-v$ in V with
  $$v + (-v) = (-v) + v = 0$$

Axioms (essential properties) of scalar multiplication

- $r(u + v) = ru + rv$ for all u, v in V and any r in $\mathbb{R}$.
- $(r + s)v = rv + sv$ for all v in V and any r, s in $\mathbb{R}$.
- $r(sv) = (rs)v$ for all v in V and any r, s in $\mathbb{R}$.
- There is an element 1 such that $1v = v$ for all v in V.

An axiom is a fact that can't be reduced to a simpler property.

Two examples of vector spaces: $\mathbb{R}^n$ and $\mathbb{P}$

The set of vectors of height n is a vector space!

Fact (The motivating examples of vector spaces)

For each positive integer n, the set $\mathbb{R}^n$ is a vector space.

Fact (Our first non-vector vector space)

The set of polynomials in x is a vector space, denoted $\mathbb{P}$.

Useful fact: Two polynomials are equal if and only if they have the coefficients when written in standard form: $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$.

- **Exercise:** Determine whether $(x-4)^3$ is a **scalar multiple** of $x^2 + x + 1$.

Exercise

Write x^2 as a **linear combination** of 1, $1+x$, and $1+2x+x^2$.
(Note: Numbers like 0, 1, and 7 count as **constant** polynomials!)

Definition: The degree of a polynomial

The **degree** of a non-zero polynomial in x is the largest power of x with non-zero coefficient.

We define $\deg(0) := -\infty$, mostly to avoid an extra case.

Fact (Polynomials of degree at most n)

For each positive integer n, the set of polynomials in x of degree at most n is a vector space, denoted $\mathbb{P}_n$.

Examples:

- $\mathbb{P}_1$ consists of polynomials $ax + b$, for a, b in $\mathbb{R}$.
- The three polynomials $(x-1)^3$, $x^2 + 3x$, and 2 are in $\mathbb{P}_3$, but the polynomials x^4 and $x^8 - 2x^3$ are not.
- $\mathbb{P}_0$ is just the constant polynomials like 0, 1, and 7, which are the same as numbers, so $\mathbb{P}_0 = \mathbb{R}$.

A non-example: Consider the set of polynomials of degree exactly 3.

By a **sequence**, we mean an infinite list of real numbers.

Examples of sequences

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots \qquad \text{(the Fibonacci sequence)}$$
$$2, 3, 5, 9, 11, 13, 17, \ldots \qquad \text{(prime numbers)}$$
$$1, 3, 9, 27, 81, 243, \ldots \qquad \text{(powers of 3)}$$
$$7, 12, -5, \pi, 3.5, 7, \ldots \qquad \text{(Just some random numbers)}$$

Fact (The set of sequences is a vector space)

The set of sequences is a vector space, denote $\mathbb{S}$. Addition and scalar multiplication are defined term-wise.

Fact (Sets of matrices of fixed size are vector spaces)

For positive integers m and n, the set of $m \times n$-matrices is a vector space, denoted $\mathbb{R}^{m\times n}$.

Addition and scalar multiplication are the matrix versions.

Example: Let's say we know $\mathbb{P}$ is a vector space, but not $\mathbb{P}_3$.

- To add two polynomials in $\mathbb{P}_3$, add them as polynomials in $\mathbb{P}$, and observe the result is still in $\mathbb{P}_3$.
- Scalar multiplication also does not leave $\mathbb{P}_3$.

Since the axioms hold in $\mathbb{P}$, they automatically hold in $\mathbb{P}_3$. **So $\mathbb{P}_3$ is a vector space.**

Definition: Subspace of a vector space

Let V be a vector space. A **subspace** of V is a non-empty subset W of V which is...

- **closed under addition**; that is,

$$\text{for all } \mathrm{v}, \mathrm{w} \text{ in } W, \text{ the sum } \mathrm{v} + \mathrm{w} \text{ is in } W, \text{ and}$$

- **closed under scalar multiplication**; that is,

$$\text{for all } \mathrm{v} \text{ in } W \text{ and } c \text{ in } \mathbb{R}, \text{ the product } c\mathrm{v} \text{ is in } W.$$

Fact (Subspaces are vector spaces)

A subspace of a vector space is also a vector space.

Exercise

Let S denote the set of polynomials in $\mathbb{P}_2$ such that $f(5) = 0$. That is,

$$S = \{f(x) \text{ in } \mathbb{P}_2 \mid f(5) = 0\}.$$

Show whether S is a subspace or not a subspace of $\mathbb{P}_2$.

Exercise

Let T denote the set of polynomials in $\mathbb{P}_2$ such that $f(5) = 1$. That is,

$$T = \{f(x) \text{ in } \mathbb{P}_2 \mid f(5) = 1\}.$$

Show whether T is a subspace or not a subspace of $\mathbb{P}_2$.

III. A subspace spanned by a set

Definition: Span$\{v_1, v_2, ..., v_p\}$

The **span** of a set of objects is the set of their linear combinations.

Example:

$$\text{Span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-10\\0\\-4\end{bmatrix}\right\} := \left\{t\begin{bmatrix}1\\1\\0\end{bmatrix} + s\begin{bmatrix}-10\\0\\-4\end{bmatrix} \text{ for all } t, s \text{ in } \mathbb{R}\right\}$$

$$\text{Span}\left\{\begin{bmatrix}1\\4\\7\end{bmatrix}, \begin{bmatrix}2\\5\\8\end{bmatrix}, \begin{bmatrix}3\\6\\9\end{bmatrix}\right\} := \left\{r\begin{bmatrix}1\\4\\7\end{bmatrix} + s\begin{bmatrix}2\\5\\8\end{bmatrix} + t\begin{bmatrix}3\\6\\9\end{bmatrix} \text{ for all } r, s, t \text{ in } \mathbb{R}\right\}$$

Theorem: Spans are subspaces

The span of a set of objects in a vector space V is a subspace of V.

- Span$\{v_1, v_2, ..., v_p\}$ is called **the subspace** spanned by $\{v_1, v_2, ..., v_p\}$.

Definition: Spanning sets

A **spanning set** (or **generating set**) of H is a set of objects whose span is H.

- If $H = \text{Span}\{v_1, v_2, ..., v_p\}$, then $\{v_1, v_2, ..., v_p\}$ is a **spanning set** for H.

Exercise

Let W be the subset of $\mathbb{R}^3$ consisting of vectors whose second entry is the average of the other two. Show that W is a subspace of $\mathbb{R}^3$.