Math 2220 Linear Algebra Sec 1.9 The matrix of a linear transformation

Main Thm: Every linear map from $\mathbb{R}^n \to \mathbb{R}^m$ can be described by a matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$.

1 The standard matrix for a linear transformation

To find this matrix A, it's enough to look at the columns of the $n \times n$ ______________________

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

1.1 Example

Suppose we know that T is a linear transformation $\mathbb{R}^3 \to \mathbb{R}^2$ such that

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, T(\mathbf{e}_1) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, T(\mathbf{e}_3) = \begin{bmatrix} 5 \\ 6 \end{bmatrix}.$$

With no additional information, find a formula for the image $T(\mathbf{x})$ for an arbitrary $\mathbf{x}$ in $\mathbb{R}^3$.

Sol: Write $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} +$ $\qquad\qquad\qquad\qquad$ $= x_1\mathbf{e}_1 +$

Then we have

$T(\mathbf{x}) = T(x_1\mathbf{e}_1 +$

1.2 Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then there is a unique __________ matrix A such that

$$T(\mathbf{x}) = \_\_\_\_\_ \text{for all } \mathbf{x} \text{ in } \_\_\_\_\_$$

This unique matrix is the concatenation $A = [T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ ...\ T(\mathbf{e}_n)]$

1.3 Example

We know T is a linear transformation $R^3 \to \mathbb{R}^2$ such that $T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, T(\mathbf{e}_1) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, T(\mathbf{e}_3) = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.

Find the standard matrix for T.

Sol: $A =$

2 Geometric linear maps of $\mathbb{R}^2$

2.1 Rotation

The map $R : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates each point in $\mathbb{R}^2$ about the origin through an angle (_________ for a positive θ) is a linear map. What is the standard matrix A of this rotation map R?

Sol:

means $R\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$

means $R\left(\begin{bmatrix}0\\1\end{bmatrix}\right) =$

2.2 Reflection

The map $F : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects each point in R^2 through a line ℓ (that passes through the origin) is a linear map.

Ex: The standard matrix for the reflection through the line $y = x$ is ...

2.3 Projection

The mapping $P : \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} \mapsto \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}\ \\ \ \\ \ \end{bmatrix}$$

is called a ________________ because P projects points in $\mathbb{R}^3$ onto the xy-plane.

3 Existence and uniqueness questions: surjectivity

<u>Definition</u> A map $T : \mathbb{R}^n \to \mathbb{R}^m$ is called ________ or ________________ if ...

Cartoon:

3.1 Examples

3.1.1 Not onto

The projection $P : \mathbb{R}^3 \to \mathbb{R}^3$ given by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ __________ onto.

<u>Why?</u>

For example,

$$\mathbf{b} = \begin{bmatrix} \\ \\ \\ \end{bmatrix} \text{\_\_\_\_\_ in the \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ } \mathbb{R}^3,$$

but $\mathbf{b}$ ____________ the image $P(\mathbf{x})$ of any $\mathbf{x}$ in the ____________ R^3.

3.1.2 Onto

However, The projection $P' : \mathbb{R}^3 \to \mathbb{R}^2$ given by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ _____ onto.

<u>Why?</u>

For any $\begin{bmatrix} x \\ y \end{bmatrix}$ in the codomain _____, we have $P'\left(\begin{bmatrix} \\ \\ \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$, so every vector $\begin{bmatrix} x \\ y \end{bmatrix}$ is the codomain $\mathbb{R}^2$ _____ in the __________ of P'.

3.1.3 Other examples

What about the linear transformation whose standard matrix is the following?

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

In general, we can answer this existence question by looking at an echelon form of the standard matrix A of the linear map T.

3.2 Theorem

Let A be an $m \times n$ matrix. Then the following are equivalent (TFAE):

I. For each $\mathbf{b} \in \mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ ________________________

II. The columns of A __________ $\mathbb{R}^m$

III. A has a pivot in every _________

IV. The matrix transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{x} \mapsto A\mathbf{x}$

3.2.1 Example

If T is the linear transformation whose standard matrix can be row-reduced to an echelon form

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 0 & 8 \end{bmatrix},$$

then T _____ onto.

Why?

3.2.2 Example

If T is the linear transformation whose standard matrix can be row-reduced to an echelon form

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

then T _________ onto.

Why?

3.2.3 Exercise

(a) Write an echelon form of the standard matrix of an non-onto linear transformation.

(b) Write an echelon form of the standard matrix of an onto linear transformation.

4 Existence and uniqueness questions: injectivity

Definition A map $T : \mathbb{R}^n \to \mathbb{R}^m$ is called ________________ or _____________ if ...

Cartoon:

Ex: The projection $P : \mathbb{R}^3 \to \mathbb{R}^3$ given by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ _________ one-to-one.

Why?

For example,

4.1 Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then T is ________________ if and only if the equation $T(\mathbf{x}) = \mathbf{0}$

Ex: The projection $P' : \mathbb{R}^3 \to \mathbb{R}^2$ given by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \quad \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ ____________ 1-to-1.

Why?

$P'\left(\begin{bmatrix} \quad \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. So $T(\mathbf{x}) = \mathbf{0}$ _______ solutions other than the _________ solution.

4.2 Theorem

Let A be a matrix. Then the following are equivalent (TFAE):

I. The equation $A\mathbf{x} = \mathbf{0}$ ...

II. The columns of A are

III. # of pivots of $A =$

IV. The matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ is

4.2.1 Example

If T is the linear transformation whose standard matrix can be row-reduced to an echelon form

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 0 & 8 \end{bmatrix},$$

then T __________ one-to-one.

Why? There is ______ free variable for the solution for $A\mathbf{x} = \mathbf{0}$, so $A\mathbf{x} = \mathbf{0}$ has

4.2.2 Example

If T is the linear transformation whose standard matrix can be row-reduced to an echelon form

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

then T ___________ one-to-one.

Why? The solution set of $A\mathbf{x} = \mathbf{0}$ has ______ free variables, so the columns of A are ______ linearly independent.

4.2.3 Example

If T is the linear transformation whose standard matrix can be row-reduced to an echelon form

$$\begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 0 & 0 \end{bmatrix},$$

then T _____ one-to-one.

Why?

By inspection, we see that the columns of A ________ linearly independent because