Recommended Problems Section 9.4

#1 Determine whether $\sum_{n=1}^{\infty} \frac{7}{5+2^n}$ converges or diverges.

#2 Determine whether $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^8+6}}$ converges or diverges.

#3 Use the Limit Comparison Test to determine whether $\sum_{n=18}^{\infty} \frac{8n^3-9n^2+18}{9+2n^4}$ converges or diverges.

#4 For each of the following series, state whether Direct Comparison Test or Limit Comparison Test can be used to check for convergence. If it is possible, apply a comparison test.

1.) $\sum_{n=1}^{\infty} \frac{(\ln(n))^4}{n+9}$

2) $\sum_{n=1}^{\infty} \frac{\cos(n)\sqrt{n}}{5n+3}$

3) $\sum_{n=1}^{\infty} \frac{n^2}{n^3+3}$

4.) $\sum_{n=1}^{\infty} \frac{(\cos(n))^2\sqrt{n}}{n^2}$

5.) $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n}$

#5 For each of the following series, determine whether it converges / diverges.

1. $\sum_{n=2}^{\infty} \frac{1}{n^7 - 36}$

2. $\sum_{n=1}^{\infty} \frac{1}{n(n+4)}$

3. $\sum_{n=1}^{\infty} \frac{9+2^n}{9+4^n}$

4. $\sum \frac{\ln n}{7n}$

5. $\sum_{n=1}^{\infty} \frac{1}{5+\sqrt[4]{n^4}}$

#6

Each of the following statements is an attempt to show that a given series is convergent or divergent not using the Comparison Test (NOT the Limit Comparison Test.) For each statement, enter C (for "correct") if the argument is valid, or enter I (for "incorrect") if any part of the argument is flawed. (Note: if the conclusion is true but the argument that led to it was wrong, you must enter I.)

☐ **1.** For all $n > 2$, $\frac{\sqrt{n+1}}{n} > \frac{1}{n}$, and the series $\sum \frac{1}{n}$ diverges, so by the Comparison Test, the series $\sum \frac{\sqrt{n+1}}{n}$ diverges.

☐ **2.** For all $n > 2$, $\frac{\ln(n)}{n^2} > \frac{1}{n^2}$, and the series $\sum \frac{1}{n^2}$ converges, so by the Comparison Test, the series $\sum \frac{\ln(n)}{n^2}$ converges.

☐ **3.** For all $n > 2$, $\frac{1}{n^2-5} < \frac{1}{n^2}$, and the series $\sum \frac{1}{n^2}$ converges, so by the Comparison Test, the series $\sum \frac{1}{n^2-5}$ converges.

☐ **4.** For all $n > 1$, $\frac{1}{n\ln(n)} < \frac{2}{n}$, and the series $2\sum \frac{1}{n}$ diverges, so by the Comparison Test, the series $\sum \frac{1}{n\ln(n)}$ diverges.

☐ **5.** For all $n > 1$, $\frac{n}{2-n^3} < \frac{1}{n^2}$, and the series $\sum \frac{1}{n^2}$ converges, so by the Comparison Test, the series $\sum \frac{n}{2-n^3}$ converges.

☐ **6.** For all $n > 1$, $\frac{\arctan(n)}{n^3} < \frac{\pi}{2n^3}$, and the series $\frac{\pi}{2}\sum \frac{1}{n^3}$ converges, so by the Comparison Test, the series $\sum \frac{\arctan(n)}{n^3}$ converges.

Recommended Problems Sec 9.4 Solutions

 # 1 Determine whether $\sum_{n=1}^{\infty} \frac{7}{5+2^n}$ converges or diverges.

Sol:

Let $a_n := \frac{7}{5+2^n}$

The term looks like $\sum \frac{1}{2^n} = \sum \left(\frac{1}{2}\right)^n$, a $\underbrace{\text{convergent}}_{\text{since ratio} < 1}$ geometric series,

(i) Using Limit Comparison Test: so try $b_n = \frac{1}{2^n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{7}{5+2^n} \cdot 2^n \underset{\text{L'H } \frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{7 \ln(2)\ 2^n}{\ln(2)\ 2^n} = 7$$

Since $\lim_{n\to\infty} \frac{a_n}{b_n}$ is a finite number 7 and $7 > 0$,

either both $\sum a_n$ and $\sum b_n$ converge

or both $\sum a_n$ and $\sum a_n$ diverge.

Since $\sum b_n$ converges, $\sum a_n$ also $\boxed{\text{converges}}$

(ii) Using Comparison Test also works with $b_n = 7\frac{1}{2^n}$

since $a_n \leq b_n$ for $n = 1, 2, \ldots$

#2 Determine whether $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^8+6}}$ converges or diverges.

Sol: We will use the Limit Comparison Test with

$$b_n = \frac{n}{\sqrt{n^8}} = \frac{n}{n^{\frac{8}{2}}} = \frac{n}{n^4} = \frac{1}{n^3}$$

Let $a_n := \frac{n}{\sqrt{n^8+6}}$

Then
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^8+6}} \cdot n^3$$

$$= \lim_{n\to\infty} \frac{n^4}{\sqrt{n^8+6}}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{n^4}{n^4}\right)}{\sqrt{\frac{n^8}{n^8}+\frac{6}{n^8}}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{6}{n^8}}} = 1$$

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, a positive number,

either $\sum a_n$ and $\sum b_n$ both converge or both diverge.

Since $\sum b_n = \sum \frac{1}{n^3}$ is a convergent p-series,

$\sum a_n$ also $\boxed{\text{converges}}$ ✓

#3 Use the Limit Comparison Test to determine whether $\sum_{n=18}^{\infty} \frac{8n^3 - 9n^2 + 18}{9 + 2n^4}$ converges or diverges.

Sol:

Step a: Let $b_n = \frac{n^3}{n^4} = \frac{1}{n}$

Step b: $\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{8n^3 - 9n^2 + 18}{9 + 2n^4} \cdot n$

$$= \lim_{n \to \infty} \frac{8n^4 - 9n^3 + 18n}{9 + 2n^4}$$

$$= \lim_{n \to \infty} \frac{\left(8\frac{n^4}{n^4} - 9\frac{n^3}{n^4} + 18\frac{n}{n^4}\right)}{\left(\frac{9}{n^4} + \frac{2n^4}{n^4}\right)}$$

$$= \lim_{n \to \infty} \frac{\left(8 - \frac{9}{n} + \frac{18}{n^3}\right)}{\left(\frac{9}{n^4} + 2\right)}$$

$$= 4$$

Step c: By the Limit Comparison Test, the series also diverges since $\sum b_n = \sum \frac{1}{n}$ is a p-series with $p = 1$ which we know is divergent.

#4 For each of the following series, state whether Direct Comparison Test or Limit Comparison Test can be used to check for convergence. If it is possible, apply a comparison test.

1. $\sum_{n=1}^{\infty} \frac{(\ln(n))^4}{n+9}$ Yes. The series diverges. Limit comparison test with $b_n = \frac{(\ln(n))^4}{n}$

2. $\sum_{n=1}^{\infty} \frac{\cos(n)\sqrt{n}}{5n+3}$ No. Comparison Tests (Sec 9.4) do not work because the sequence is sometimes negative, sometimes positive

3. $\sum_{n=1}^{\infty} \frac{n^2}{n^3+3}$ Yes. The series diverges. Limit comparison Test with $b_n = \frac{1}{n}$

4. $\sum_{n=1}^{\infty} \frac{(\cos(n))^2\sqrt{n}}{n^2}$ Yes. The series converges. Limit Comparison Test / Comparison Test with $b_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$

5. $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n}$ No. Comparison Tests (Sec 9.4) do not work because the sequence is sometimes negative, sometimes positive

#5 For each of the following series, determine whether it converges / diverges.

1. $\sum_{n=2}^{\infty} \frac{1}{n^7-36}$ Convergent by Limit Comparison Test with $b_n = \frac{1}{n^7}$

2. $\sum_{n=1}^{\infty} \frac{1}{n(n+4)}$ Converges by Limit Comparison Test / Comparison Test with $b_n = \frac{1}{n^2}$

3. $\sum_{n=1}^{\infty} \frac{9+2^n}{9+4^n}$

Let $a_n = \frac{9+2^n}{9+4^n}$

Try $b_n = \frac{2^n}{4^n} = \frac{2^n}{2^{2n}} = \left(\frac{2}{4}\right)^n = \left(\frac{1}{2}\right)^n$

$$\frac{a_n}{b_n} = \frac{9+2^n}{9+4^n} \cdot 2^n = \frac{2^n + 2^n 2^n}{9+4^n} = \frac{\frac{1}{4^n}\left[2^n + 2^n 2^n\right]}{\frac{1}{4^n}\left[9+4^n\right]} = \frac{\frac{2^n}{4^n}+1}{\frac{9}{4^n}+1} = \frac{\left(\frac{1}{2}\right)^n+1}{\frac{9}{4^n}+1}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{1}{2}\right)^n+1}{\frac{9}{4^n}+1} = \boxed{1},$$ which is a positive number,

so either $\sum a_n$ and $\sum b_n$ both converge, or $\sum a_n$ and $\sum b_n$ both diverge.

We know $\sum \left(\frac{1}{2}\right)^n$ is a convergent geometric series (since $\frac{1}{2}$ is in $(-1,1)$), so $\sum a_n$ also converges.

4. $\sum \frac{\ln n}{7n}$ diverges by Comparison Test with $b_n = \frac{1}{n}$

5. $\sum_{n=1}^{\infty} \frac{1}{5+\sqrt[4]{n^4}}$ diverges by Comparison Test with $b_n = \frac{1}{n}$

#6

Each of the following statements is an attempt to show that a given series is convergent or divergent not using the Comparison Test (NOT the Limit Comparison Test.) For each statement, enter C (for "correct") if the argument is valid, or enter I (for "incorrect") if any part of the argument is flawed. (Note: if the conclusion is true but the argument that led to it was wrong, you must enter I.)

1. For all $n > 2$, $\frac{\sqrt{n+1}}{n} > \frac{1}{n}$, and the series $\sum \frac{1}{n}$ diverges, so by the Comparison Test, the series $\sum \frac{\sqrt{n+1}}{n}$ diverges.

2. For all $n > 2$, $\frac{\ln(n)}{n^2} > \frac{1}{n^2}$, and the series $\sum \frac{1}{n^2}$ converges, so by the Comparison Test, the series $\sum \frac{\ln(n)}{n^2}$ converges.

3. For all $n > 2$, $\frac{1}{n^2-5} < \frac{1}{n^2}$, and the series $\sum \frac{1}{n^2}$ converges, so by the Comparison Test, the series $\sum \frac{1}{n^2-5}$ converges.

4. For all $n > 1$, $\frac{1}{n\ln(n)} < \frac{2}{n}$, and the series $2\sum \frac{1}{n}$ diverges, so by the Comparison Test, the series $\sum \frac{1}{n\ln(n)}$ diverges.

5. For all $n > 1$, $\frac{n}{2-n^3} < \frac{1}{n^2}$, and the series $\sum \frac{1}{n^2}$ converges, so by the Comparison Test, the series $\sum \frac{n}{2-n^3}$ converges.

6. For all $n > 1$, $\frac{\arctan(n)}{n^3} < \frac{\pi}{2n^3}$, and the series $\frac{\pi}{2}\sum \frac{1}{n^3}$ converges, so by the Comparison Test, the series $\sum \frac{\arctan(n)}{n^3}$ converges.

Sol:

(1 point) **Library/Rochester/setSeries6CompTests/benny_ser3B.pg**

Each of the following statements is an attempt to show that a given series is convergent or divergent not using the Comparison Test (NOT the Limit Comparison Test.) For each statement, enter C (for "correct") if the argument is valid, or enter I (for "incorrect") if any part of the argument is flawed. (Note: if the conclusion is true but the argument that led to it was wrong, you must enter I.)

C **1.** For all $n > 2$, $\frac{\sqrt{n+1}}{n} > \frac{1}{n}$, and the series $\sum \frac{1}{n}$ diverges, so by the Comparison Test, the series $\sum \frac{\sqrt{n+1}}{n}$ diverges.

Incorrect **2.** For all $n > 2$, $\frac{\ln(n)}{n^2} > \frac{1}{n^2}$, and the series $\sum \frac{1}{n^2}$ converges, so by the Comparison Test, the series $\sum \frac{\ln(n)}{n^2}$ converges.

Not true

3. For all $n > 2$, $\frac{1}{n^2-5} < \frac{1}{n^2}$, and the series $\sum \frac{1}{n^2}$ converges, so by the Comparison Test, the series $\sum \frac{1}{n^2-5}$ converges.

Incorrect **4.** For all $n > 1$, $\frac{1}{n\ln(n)} < \frac{2}{n}$, and the series $2\sum \frac{1}{n}$ diverges, so by the Comparison Test, the series $\sum \frac{1}{n\ln(n)}$ diverges.

Incorrect **5.** For all $n > 1$, $\frac{n}{2-n^3} < \frac{1}{n^2}$, and the series $\sum \frac{1}{n^2}$ converges, so by the Comparison Test, the series $\sum \frac{n}{2-n^3}$ converges.

C **6.** For all $n > 1$, $\frac{\arctan(n)}{n^3} < \frac{\pi}{2n^3}$, and the series $\frac{\pi}{2}\sum \frac{1}{n^3}$ converges, so by the Comparison Test, the series $\sum \frac{\arctan(n)}{n^3}$ converges.

Note: #5 is incorrect because Comparison Test can only be applied to series w/ positive terms