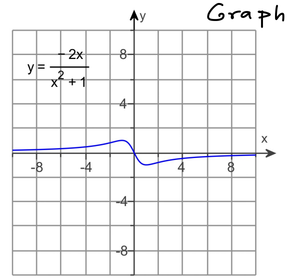
Day 1: Review Ch 3 & 4

*Review Sec 3.1 Tangent Lines
3.3 Quotient Rule

Ex: Find the tangent lines to $y = -\frac{2x}{x^2+1}$ at the origin and at the point $(1,-1)$.

Graph:



Answer : $y = -2\left(\frac{x}{x^2+1}\right)$

$$y' = -2\left[\frac{(x^2+1)\,1 - x\,(2x)}{(x^2+1)^2}\right]$$

$$= -2\left[\frac{x^2+1-2x^2}{(x^2+1)^2}\right]$$

$$= -2\left[\frac{-x^2+1}{(x^2+1)^2}\right]$$

$$= \frac{2\,(x^2-1)}{(x^2+1)^2}$$

Quotient Rule

Low D High − High D Low
over the square of
what's below

• Slope of tangent line at $\underbrace{(0,0)}_{(x_0,\,y_0)}$ is $y'(0) = 2\frac{(-1)}{1} = -2$

Tangent line: $\frac{y-y_0}{x-x_0} = -2$

$$\boxed{y = -2x}$$

confidence check

• Slope of tangent line at $(1,-1)$ is $y'(1) = 2\frac{(0)}{1} = 0$

Tangent line:
The horizontal line passing through the point $(1,-1)$:

$$\boxed{y = -1}$$

confidence check

* Review Sec 3.9 Inverse trig fcns

Ex: Use inverse function relationship & implicit differentiation to differentiate $y = \arccos(6x^3) = \cos^{-1}(6x^3)$

same thing: inverse of cosine

Answer:

① Apply cos to both sides:

$$\cos(y) = \cos(\arccos(6x^3))$$
$$\cos(y) = 6x^3$$

② Differentiate both sides (implicit differentiation):

$$\frac{d}{dx}(\cos(y)) = \frac{d}{dx}(6x^3)$$

Chain Rule

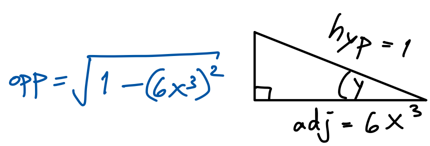
$$-\sin(y)\,\frac{dy}{dx} = 6 \cdot 3x^2$$

③ Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{18x^2}{\sin(y)}$$

④ Express in terms of x:

$\cos(y) = \dfrac{6x^3}{1}$ from step ①



$$\cos(y) = \frac{\text{adj}}{\text{hyp}}$$

So $\sin(y) = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{\sqrt{1-(6x^3)^2}}{1}$

So $\boxed{\dfrac{dy}{dx} = \dfrac{-18x^2}{\sqrt{1-36x^6}}}$

* Review Sec 4.5 Indeterminate forms & l'Hôpital's Rule

Ex: Evaluate $\lim\limits_{x \to 0} \dfrac{x^2}{\cos(x) - 1}$

Answer: $\lim\limits_{x \to 0} x^2 = 0$

$\lim\limits_{x \to 0} \cos(x) - 1 = \cos(0) - 1 = 0$

We have an indeterminate form "$\frac{0}{0}$".

We can try applying l'Hôpital's Rule.

$$\lim_{x \to 0} \frac{x^2}{\cos x - 1} \underset{\text{l'H}}{=} \lim_{x \to 0} \frac{2x}{-\sin x} \underset{\text{l'H again}}{=} \lim_{x \to 0} \frac{2}{-\cos x} \underset{\substack{\text{Sec 2.2} \\ \text{Limit Law} \\ \text{for quotient}}}{=} \frac{\lim\limits_{x \to 0} 2}{\lim\limits_{x \to 0} -\cos x} = \frac{2}{-1} = \boxed{-2}$$

$2x \to 0$ as $x \to 0$

$-\sin x \to 0$ as $x \to 0$

* Review

Sec 3.5 Derivatives of trig funs

3.6 Chain Rule

3.7 Implicit Diff

3.3 Product Rule

$\frac{d}{dx}(\sin x) = \cos x$ $\quad$ $\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = (\sec x)^2$ $\quad$ $\boxed{\frac{d}{dx}(\cot x) = -(\csc x)^2}$

$\frac{d}{dx}(\sec x) = \sec x \tan x$ $\quad$ $\frac{d}{dx}(\csc x) = -\csc x \cot x$

$\csc x = \frac{1}{\sin x}$

Ex: Given $x + \cot(xy) = 2$, perform implicit differentiation to find $\frac{dy}{dx}$

Answer: Differentiate both sides term by term

$$\underbrace{\frac{d}{dx} x}_{1} + \underbrace{\frac{d}{dx}\left(\cot(xy)\right)} = \underbrace{\frac{d}{dx}(2)}_{0}$$

$$\frac{d}{dx}\cot(xy) \overset{\text{Chain Rule}}{=} -(\csc(xy))^2 \underbrace{\frac{d}{dx}(xy)}$$

$$\frac{d}{dx}(xy) \overset{\text{product Rule}}{=} x\frac{dy}{dx} + 1\,y$$

$$\text{So } \frac{d}{dx}\cot(xy) = -(\csc(xy))^2\left(x\frac{dy}{dx} + y\right)$$

$$= -x(\csc(xy))^2\frac{dy}{dx} - y(\csc(xy))^2$$

$$1 + -x(\csc(xy))^2\frac{dy}{dx} - y(\csc(xy))^2 = 0$$

$$1 - (y\csc(xy))^2 = x(\csc(xy))^2\frac{dy}{dx}$$

$$\frac{1 - y(\csc(xy))^2}{x(\csc(xy))^2} = \frac{dy}{dx}$$

$$\boxed{\frac{(\sin(xy))^2 - y\,1}{x} = \frac{dy}{dx}}$$