9.9 Convergence of Taylor Series

Recall from Sec 9.8 that the Maclaurin series of the function $f(x) = e^x$ is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ [From Sec 9.7, the interval of convergence is $(-\infty, \infty)$.]

Example:

Find $\overbrace{\text{the Taylor series at } x=0}^{\text{the Maclaurin series}}$ for the functions ...

a.) e^{-9x}

b.) $18 \times e^{-9x}$

Sol

a.) If $f(x) = e^x$ then $f(-9x) = \overbrace{e^{-9x}}^{\text{our given function for part (a)}}$

So $e^{-9x} = f(-9x) = \sum_{n=0}^{\infty} \frac{(-9x)^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 9^n}{n!} x^n}$

b.) If $f(x) = e^x$ then $18 \times f(-9x) = \overbrace{18 \times e^{-9x}}^{\text{our given function for part (b)}}$

So $18 \times e^{-9x} = 18 \times f(-9x) = 18 \times \overbrace{\sum_{n=0}^{\infty} \frac{(-1)^n 9^n}{n!} x^n}^{\text{from part (a)}}$

$$= \sum_{n=0}^{\infty} 18 \times (-1)^n 9^n x^n \frac{1}{n!}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{18 (-1)^n 9^n}{n!} x^{n+1}}$$

Note: Interval of convergence is $(-\infty, \infty)$ for both

Recall from Sec 9.8 that the Maclaurin series of the function $f(x) = \cos x$ is $1x^0 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \dots$

(Reindex so that $n=0 \quad n=1 \quad n=2 \quad n=3 \quad n=4$)

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

[Check using Ratio Test that the interval of convergence is $(-\infty, \infty)$.]

Example:

Find $\underbrace{\text{the Taylor series at } x=0}_{\text{the Maclaurin series}}$ for the functions ...

a.) $x^2 \cos \frac{\pi x}{2}$

b.) $e^x \cos x$ (just the first four nonzero terms)

Sol a) If $g(x) = \cos x$ then $x^2 g\left(\frac{\pi x}{2}\right) = x^2 \cos \frac{\pi x}{2}$

so $x^2 \cos\left(\frac{\pi x}{2}\right) = x^2 g\left(\frac{\pi x}{2}\right) = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\pi x}{2}\right)^{2n} = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\pi}{2}\right)^{2n} x^{2n}$

$$= \sum_{n=0}^{\infty} x^2 \frac{(-1)^n}{(2n)!}\left(\frac{\pi}{2}\right)^{2n} x^{2n} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\pi}{2}\right)^{2n} x^{2n+2}}$$

b) If $f(x) = e^x$ and $g(x) = \cos x$ then $f(x)\,g(x) = e^x \cos x$

So $e^x \cos x = f(x)\,g(x) = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}\right) =$

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$$

Multiply the first series by each term of the second series.

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) - \left(\frac{x^2}{2!} + \frac{x^3}{2!} + \frac{x^4}{2!2!} + \frac{x^5}{2!3!} + \cdots\right) + \left(\frac{x^4}{4!} + \frac{x^5}{4!} + \frac{x^6}{2!4!} + \cdots\right) + \cdots$$

$$= \boxed{1 + x - \frac{x^3}{3} - \frac{x^4}{6}} + \cdots$$

The first four nonzero terms

Note: Interval of convergence is $(-\infty, \infty)$ for both

Example:

a) Find the $\overbrace{\text{Taylor series at } x=0}^{\text{the Maclaurin series}}$ for the function $\frac{1}{3}(2x + x\cos x)$

b) Find the first three nonzero terms

Sol

a) $$\frac{1}{3}(2x + x\cos x) = \frac{2}{3}x + \frac{1}{3}x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k\frac{x^{2k}}{(2k)!} + \cdots\right)$$ Taylor series for $\cos x$

$$= \underbrace{\frac{2}{3}x + \frac{1}{3}x - \frac{x^3}{3!} + \frac{x^5}{3\cdot 4!} - \cdots}_{} = \boxed{x - \frac{x^3}{6} + \frac{x^5}{72}} - \cdots$$

$$= \frac{2}{3}x + \sum_{n=0}^{\infty} \frac{1}{3}x \, \frac{(-1)^n x^{2n}}{(2n)!}$$

$$= \boxed{\frac{2}{3}x + \sum_{n=0}^{\infty} \frac{1}{3}\frac{(-1)^n}{(2n)!} x^{2n+1}}$$

b) The first three nonzero terms

Review of Sec 9.7 Part B

Example: Find a power series representation for $f(x) = \dfrac{2x^4}{2-3x}$ and find its interval of convergence.

$$\frac{\frac{1}{2}\,2x^4}{\frac{1}{2}(2-3x)} = \frac{x^4}{1-\frac{3}{2}x}$$

$$= x^4 \frac{1}{1-\left(\frac{3}{2}x\right)}$$

Radius of convergence

$$= x^4 \sum_{n=0}^{\infty} \left(\frac{3}{2}x\right)^n \quad \text{if } \left|\frac{3}{2}x\right| < 1 \iff |x| < \frac{2}{3}$$

$$= x^4 \sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n x^n$$

$$= \boxed{\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n x^{4+n}}$$

Interval of convergence : $\left(-\frac{2}{3}, \frac{2}{3}\right)$

Example:

Find the Taylor series at $x=0$ for $f(x) = \dfrac{81}{(1-x)^2}$ and its radius of convergence.

- We know $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$ with radius of convergence $R=1$.
- Take the derivative of both sides

$$-\frac{1}{(1-x)^2}(-1) = 0 + \sum_{n=1}^{\infty} n x^{n-1}$$

Radius of convergence is still $R=1$

- So $\dfrac{81}{(1-x)^2} = \displaystyle\sum_{n=1}^{\infty} 81\, n\, x^{n-1}$.

Radius of convergence is still $R=1$