9.8 Taylor and Maclaurin Series

Suppose for $|x-a|<R$, we have

True for all "nice" functions $f(x)$

$$f(x)=c_0+c_1(x-a)+c_2(x-a)^2+c_3(x-a)^3+c_4(x-a)^4+\cdots,$$

power series centered at a with radius of convergence R

then $f(a)=c_0$. We can differentiate both sides with respect to x to get

$$f'(x)=c_1+2c_2(x-a)+3c_3(x-a)^2+4c_4(x-a)^3+\cdots,$$

then $f'(a)=c_1$. Again, we have

$$f''(x)=2c_2+3(2)c_3(x-a)+4(3)c_4(x-a)^2+5(4)c_5(x-a)^3+\ldots,$$

then $f''(a)=2c_2$. Apply the procedure again to obtain

$$f'''(x)=3(2)c_3+4(3)(2)c_4(x-a)+5(4)(3)c_5(x-2)^2+\ldots,$$

then $f'''(a)=3(2)c_3$. Apply the procedure one more time to obtain

$$f^{(4)}(x)=4(3)(2)c_4+5(4)(3)(2)c_5(x-2)+\ldots,$$

then $f^{(4)}(a)=4(3)(2)c_4$. By now you can see the pattern. If we continue to differentiate and substitute $x=a$, we obtain

$$f^{(n)}(a)=n(n-1)(n-2)\cdots(2)c_n$$
$$=n!\,c_n.$$

So

$$\boxed{c_n=\frac{f^{(n)}(a)}{n!}.}$$

Example:

If $f(x)=\sum_{n=0}^{\infty}c_nx^n$ and $f(0)=14$, $f'(0)=-15$, $f''(0)=-1$, $f'''(0)=-1$,

find the first four terms of $\sum_{n=0}^{\infty}c_nx^n$.

Answer $c_0=\frac{f(0)}{0!}=14$ $\quad c_1=\frac{f'(0)}{1!}=-15$ $\quad c_2=\frac{f''(0)}{2!}=-\frac{1}{2}$ $\quad c_3=\frac{f'''(0)}{3!}=-\frac{1}{6}$

$$\boxed{14-15x-\frac{1}{2}x^2-\frac{1}{6}x^3}$$

Polynomial approximations:

DEFINITIONS Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$** is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The **Maclaurin series of f** is the Taylor series generated by f at $x = 0$, or

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots.$$

Example: Find the Maclaurin series of the function $f(x) = e^x$

Solution Since $f^{(n)}(x) = e^x$ and $f^{(n)}(0) = 1$ for every $n = 0, 1, 2, \ldots$, the Taylor series generated by f at $x = 0$ is

$$\begin{aligned} f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots \\ = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \end{aligned}$$

DEFINITION Let f be a function with derivatives of order k for $k = 1, 2, \ldots, N$ in some interval containing a as an interior point. Then, for any integer n from 0 through N, the **Taylor polynomial of order n** generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Example:

Find the 3rd Taylor polynomial for $f(x) = e^x$ at $x=0$

Sol: $P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}$

Example:

In general, the Taylor polynomial for $f(x) = e^x$ at $x=0$ of order n is

$$P_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!}$$

Example Compute the Maclaurin seris for $\cos x$.

Sol:

$f(x) = \cos x \qquad f(0) = 1$

$f'(x) = -\sin x \qquad f'(0) = 0$

$f''(x) = -\cos x \qquad f''(0) = -1$

$f'''(x) = \sin x \qquad f'''(0) = 0$

$f^{(4)}(x) = \cos x \qquad f^{(4)}(0) = 1$

$\vdots$

The pattern repeats in a cycle of four

$f^{(4n)}(0) = 1$

$f^{(4n+1)}(0) = 0$

$f^{(4n+2)}(0) = -1$

$f^{(4n+3)}(0) = 0$

The Maclaurin series is $1x^0 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \ldots$

(Reindex so that $n=0$ $n=1$ $n=2$ $n=3$ $n=4$)

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

Example

Find the 3rd Taylor polynomial for $f(x) = \sin(x)$ at $x = \frac{\pi}{3}$

Sol:

$f(x) = \sin x \qquad f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$f'(x) = \cos x \qquad f'\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$f''(x) = -\sin x \qquad f''\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$f'''(x) = -\cos x \qquad f'''\left(\frac{\pi}{3}\right) = -\frac{1}{2}$

$$P_3(x) = \frac{\sqrt{3}}{2} + \frac{\left(\frac{1}{2}\right)}{1!}\left(x - \frac{\pi}{3}\right) + \frac{\left(-\frac{\sqrt{3}}{2}\right)}{2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{\left(-\frac{1}{2}\right)}{3!}\left(x - \frac{\pi}{3}\right)^3$$

$$\boxed{P_3(x) = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right) + \frac{-\sqrt{3}}{4}\left(x - \frac{\pi}{3}\right)^2 - \frac{1}{12}\left(x - \frac{\pi}{3}\right)^3}$$