9.6 Alternating Series and Conditional Convergence

A series is called alternating if the terms are alternately positive and negative.

Thm

Alternating Series Test.

IF the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\, b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots, \qquad \text{where all } b_n > 0$$

or

$$\sum_{n=1}^{\infty}(-1)^{n}\, b_n = -b_1 + b_2 - b_3 + b_4 - b_5 + b_6 - \cdots, \qquad \text{where all } b_n > 0$$

satisfies

(i) $b_{n+1} \leq b_n$ for all n, (eventually $\{b_n\}$ is decreasing)

for $n \geq N$, for some positive integer N

(ii) $\lim_{n\to\infty} b_n = 0$,

THEN

the series converges.

Note: If either condition in this test fails then the test cannot be used.

Note: • The Alternating Series Test can only be used to conclude convergence.

• If you think the series diverges, you have to use a different Test.

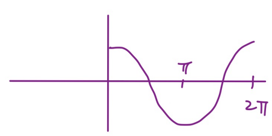
True or false

1. True or false? $\sum_{n=1}^{\infty} \frac{\cos(n)}{n}$ is an alternating series.

2. True or false? The infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^{1/4}}$ is alternating.

$-\frac{1}{1} + \frac{1}{2} + \frac{-1}{3} + \frac{1}{4} + \frac{-1}{5} + \dots$

$$\cos(n\pi) = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$



so $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^{1/4}} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^{1/4}}$ is alternating with $b_n = \frac{1}{n^{1/4}}$.

Example 1:

Determine whether $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ converges or diverges.

Sol: Since $\frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}}$ for $n = 1, 2, 3, \ldots$

and

$\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$,

this alternating series satisfies the condition for the Alternating Series Test.

So $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ <u>converges</u>

Example 2: The alternating series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^{3/2}}$ converges by the Alternating Series Test because

$\frac{1}{(n+1)^{\frac{3}{2}}} \leq \frac{1}{n}$ for $n = 1, 2, 3, \ldots$

and

$\lim_{n \to \infty} \frac{1}{n^{3/2}} = 0$

Example 3:
In general, if $p > 0$, the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ <u>converges</u> by the Alternating Series Test.

New Definition

A series is called <u>conditionally convergent</u> if it is convergent but not absolutely convergent

Recall:

- If a series is absolutely convergent, then it is convergent
- If a series is convergent, it may be absolutely convergent or not absolutely convergent.

$\sum a_n$

- $\to$ $\sum a_n$ is divergent $\to$ $\sum |a_n|$ is divergent
  - Just say $\sum a_n$ is <u>divergent</u>
- $\to$ $\sum a_n$ is convergent
  - $\to$ $\sum |a_n|$ is divergent
    - Say $\sum a_n$ is <u>conditionally</u> <u>convergent</u>
  - $\to$ $\sum |a_n|$ is convergent
    - Say $\sum a_n$ is <u>absolutely convergent</u>

Example 1 again:

$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ is conditionally convergent because

$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is convergent (by Alternating Series Test)

but

$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is divergent (because it's a p-series with $p = \frac{1}{2} < 1$)

Example 2 again:

$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^{3/2}}$ is absolutely convergent because

$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^{\frac{3}{2}}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ is convergent (because it's a p-series with $p = \frac{3}{2} > 1$)

(Note: Since the series is absolutely convergent, it is convergent.)

Example 3 again: The alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ ...

a.) ... <u>converges absolutely</u> when $1 < p$

because $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^p} \right| = \sum \frac{1}{n^p}$ is a convergent p-series

b.) ... <u>converges conditionally</u> when $0 < p \leq 1$

because it converges by the Alternating Series Test

but $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^p} \right| = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is a divergent p-series

c.) ... <u>diverges</u> when $p \leq 0$

because $\lim_{n \to \infty} \frac{(-1)^n}{n^p}$ does not exist

so it diverges by the n-th Term Test for Divergence.