Sec 9.5 Absolute Convergence & the Ratio Test

Def A series $\sum_{n=1}^{\infty} a_n$ converges absolutely (i.e. the series is absolutely convergent) if $\sum_{n=1}^{\infty} |a_n|$ converges.

Ex Is $\sum_{n=1}^{\infty} 5\left(\frac{-1}{4}\right)^n$ absolutely convergent?

Ans The corresponding sum of absolute values is $\sum_{n=1}^{\infty} \left|5\left(-\frac{1}{4}\right)^n\right| = \sum_{n=1}^{\infty} 5\left(\frac{1}{4}\right)^n$, which we know is convergent because it's a geometric series with ratio $\frac{1}{4}$ which is in $(-1, 1)$.

So the series $\sum_{n=1}^{\infty} 5\left(-\frac{1}{4}\right)^n$ is absolutely convergent.

Theorem:

If a series is absolutely convergent, then it is convergent.

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

Ex: Determine whether $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2} = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} + \ldots$ converges.

Sol: The corresponding series of absolute values is $\sum_{n=1}^{\infty} \left| (-1)^n \frac{1}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which is a convergent p-series.

By the above theorem, $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ also converges.

Ex: Determine whether $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges or diverges

Sol:

(We can apply comparison Test to $\sum \left| \frac{\sin n}{n^2} \right|$ with $\sum \frac{1}{n^2}$)

Let $a_n := \frac{|\sin n|}{n^2}$ and $b_n := \frac{1}{n^2}$

Since (i) $0 \le a_n \le b_n$ for all $n = 1, 2, 3, \ldots$ and

(ii) $\sum b_n$ is a convergent p-series ($p = 2$),

$\sum \left| \frac{\sin n}{n^2} \right|$ also converges by the Comparison Test.

By def, $\sum \frac{\sin n}{n^2}$ absolutely converges.

Sample answer to follow

The theorem tells us that $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges.

($\sum \frac{\sin n}{n^2}$ converges and $\sum \left| \frac{\sin n}{n^2} \right|$ converges).

Definition **Factorial**

The **factorial** of a positive integer n, denoted by $n!$, is the **product** of all positive integers less than or equal to n.

- Simplify $4! = $ $4 \cdot 3 \cdot 2 \cdot 1 = 24$.
- $0! \overset{def}{=} 1$.
- Simplify $\frac{(n+1)!}{n!} = \frac{(n+1)\cancel{(n)(n-1)\cdots 2\cdot 1}}{\cancel{n(n-1)\cdots 2\cdot 1}} = n+1$.

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Theorem **The Ratio Test** Memorize !

Suppose $\sum_{n=1}^{\infty} a_n$ is an infinite series with positive terms. Consider $r := \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$.

- (i) If $0 \le r < 1$, then $\sum_{n=1}^{\infty}|a_n|$ is convergent (we say " $\sum_{n=1}^{\infty} a_n$ is absolutely convergent).
- (ii) If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.
- (iii) $r = 1$, then the Ratio Test is inconclusive

Example: Use the **Ratio Test** to determine whether the series $\sum_{k=1}^{\infty}\frac{10^k}{k!}$ converge.

$$\frac{a_{k+1}}{a_k} = \frac{\left(\frac{10^{k+1}}{(k+1)!}\right)}{\left(\frac{10^k}{k!}\right)} = \frac{10^{k+1}}{(k+1)!}\cdot\frac{k!}{10^k} = \frac{10}{k+1}$$

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k} = \lim_{k\to\infty}\frac{10}{k+1} = 0$$

$\sum_{k=1}^{\infty}\frac{10^k}{k!}$ is convergent / divergent by the Ratio Test, since $\lim_{k\to\infty}\frac{a_{k+1}}{a_k} = 0 < 1$

(absolutely convergent)

Sample answer

EX: Is the series $\sum_{n=0}^{\infty} \frac{(2n)!}{n!\,n!}$ convergent ?

Sol:

Let $a_n = \frac{(2n)!}{n!\,n!}$

$$\frac{a_{n+1}}{a_n} = \frac{(2(n+1))!}{(n+1)!\,(n+1)!} \cdot \frac{n!\,n!}{(2n)!}$$

$$\boxed{\frac{n!}{(n+1)!} = \frac{\cancel{n!}}{(n+1)\cancel{n!}} = \frac{1}{n+1}}$$

$$= \frac{(2n+2)!}{(2n)!} \frac{n!}{(n+1)!} \frac{n!}{(n+1)!}$$

$$\boxed{\frac{(2n+2)!}{(2n)!} = \frac{(2n+2)(2n+1)\cancel{(2n)!}}{\cancel{(2n)!}} = (2n+2)(2n+1)}$$

$$= (2n+2)(2n+1)\,\frac{1}{n+1}\,\frac{1}{n+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = \lim_{n\to\infty} \frac{4n^2}{n^2} = 4 > 1$$

By the Ratio Test, the series <u>diverges</u>

(Note we can also use the n-th tem Test for Divergence to conclude that this series diverges :

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(2n)(2n-1)\cdots(2n+1)}{n \cdot n \cdots n} \neq 0$$

Ex: Apply the Ratio Test to $\sum_{n=1}^{\infty} \frac{4^n\, n!\, n!}{(2n)!}$

Sol: Let $a_n = \frac{4^n\, n!\, n!}{(2n)!}$

So $$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}\,(n+1)!\,(n+1)!}{(2(n+1))!} \cdot \frac{(2n)!}{4^n\, n!\, n!}$$

$$= \frac{4^{n+1}}{4^n}\ \frac{(n+1)!}{n!}\ \frac{(n+1)!}{n!}\ \frac{(2n)!}{(2n+2)!}$$

$$= 4\ (n+1)(n+1)\ \frac{1}{(2n+2)(2n+1)}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{4\,(n+1)(n+1)}{(2n+2)(2n+1)}$$

$$= \lim_{n\to\infty} \frac{4n^2}{4n^2} = 1$$

$$\boxed{\text{The Ratio Test is inconclusive}}$$

We can use the n-th term Test for Divergence:

The ratio $\frac{a_{n+1}}{a_n} = \frac{\overset{2}{\cancel{4}}\,\cancel{(n+1)}\,(n+1)}{\cancel{2}\,\cancel{(n+1)}\,(2n+1)} = \frac{2n+2}{2n+1}$ is always bigger than 1, so $\{a_n\}$ is an increasing sequence.

Every term $a_n \geq a_1 = 2$, so $\lim_{n\to\infty} a_n \neq 0$.

By the Test for Divergence, $\sum_{n=1}^{\infty} a_n$ diverges.

EXAMPLE 4 Test the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ for absolute convergence.

SOLUTION We use the Ratio Test with $a_n = (-1)^n n^3/3^n$:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\dfrac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\dfrac{(-1)^n n^3}{3^n}}\right| = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$$

$$= \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3}\left(1 + \frac{1}{n}\right)^3 \to \frac{1}{3} < 1$$

Thus, by the Ratio Test, the given series is absolutely convergent. ■