Sec 9.4 Comparison Tests

What we know so far ...

Sec 9.2
- Geometric series
- Telescoping series
- nth term Test for Divergence

Sec 9.3
- Integral ~~Test~~ (Don't need to use)
- p-series (harmonic series is a p-series)

Harmonic series $\sum \frac{1}{k}$

The Comparison Tests

Remember: The tests on this lecture can only be applied to series with positive terms.

The Comparison Test

Theorem **The Comparison Test**

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are infinite series with **positive** terms.

Usually $\sum b_n$ will be a geometric series $\sum r^n$ or p-series $\sum \frac{1}{n^p}$

- If $a_n \le b_n$ (smaller) and $\sum_{n=1}^{\infty} b_n$ **converges**, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $a_n \ge b_n$ (bigger) and $\sum_{n=1}^{\infty} b_n$ **diverges**, then $\sum_{n=1}^{\infty} a_n$ diverges.

Ex: $\sum_{n=1}^{\infty} \frac{7}{5+2^n}$ Let $a_n := \frac{7}{5+2^n}$. Try $b_n = \frac{7}{2^n} = 7\left(\frac{1}{2}\right)^n$

(i) $a_n = \frac{7}{5+2^n} \le \frac{7}{2^n} = b_n$ for all $n = 1, 2, 3, \ldots$

(ii) $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 7\left(\frac{1}{2}\right)^n$ converges because it's a geometric series with ratio $\frac{1}{2}$

So $\sum_{n=1}^{\infty} a_n$ converges by the Direct Comparison Test.

***p*-series Test (review).** The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p = 1$ or less than 1.

Ex $\sum_{n=1} \frac{\sqrt{n+1}}{n}$ is convergent or divergent?

Let $a_n = \frac{\sqrt{n+1}}{n}$. Try $b_n = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$

(i) $a_n = \frac{\sqrt{n+1}}{n} \geq \frac{\sqrt{n}}{n} = b_n$ for all $n = 1, 2, 3, \ldots$

(ii) $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is a divergent p-series $\left(p = \frac{1}{2} \le 1\right)$

So $\sum_{n=1}^{\infty} a_n$ diverges by the Direct Comparison Test.

The Comparison Tests

The Limit Comparison Test

Theorem **The Limit Comparison Test**

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are infinite series with **positive** terms. Let $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.

1. • If L is a positive *number*, then (THINK: a_n is close to $\underbrace{L}_{\text{a positive number}} b_n$ for large n)

   $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.

2. • If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ **converges**, then (THINK: a_n is much smaller than b_n for large n)

   $\sum_{n=1}^{\infty} a_n$ also converges

3. • If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ **diverges**, then (THINK: a_n is much bigger than b_n for large n)

   $\sum_{n=1}^{\infty} a_n$ also diverges

Example. Using the **Limit Comparison Test**, determine if the series $\sum_{n=1}^{\infty} \frac{n^4 - 2n^2 + 3}{2n^6 - n + 5}$ converges.

Step 0 (Brainstorm).

- Dominant term of the top function: n^4
- Dominant term of the bottom function: $2n^6$ or n^6 $\qquad \frac{n^4}{n^6} = \frac{1}{n^2}$
- So, try comparing this series with a p-series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ where p=2

Step 1. Let $a_n = \frac{n^4 - 2n^2 + 3}{2n^6 - n + 5}$, $b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^4 - 2n^2 + 3}{2n^6 - n + 5} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{n^6 - 2n^4 + 3n^2}{2n^6 - n + 5} = \lim_{n\to\infty} \frac{n^6}{2n^6} = \frac{1}{2}$$

Step 2. Since $\lim_{n\to\infty} \frac{a_n}{b_n}$ is positive, the series $\sum_{n=1}^{\infty} \frac{n^4 - 2n^2 + 3}{2n^6 - n + 5}$ converges by the Limit Comparison Test

(since $\sum b_n = \sum \frac{1}{n^2}$ is a convergent p-series)

Ex: $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ is convergent or divergent?

What b_n will work?

EXAMPLE 3 Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergence.

SOLUTION We use the Limit Comparison Test with

$$a_n = \frac{1}{2^n - 1} \qquad b_n = \frac{1}{2^n}$$

and obtain

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(2^n - 1)}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{1}{1 - 1/2^n} = 1 > 0$$

Ex: $\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ is convergent or divergent?

What b_n will work?

EXAMPLE 4 Determine whether the series $\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ converges or diverges.

SOLUTION The dominant part of the numerator is $2n^2$ and the dominant part of the denominator is $\sqrt{n^5} = n^{5/2}$. This suggests taking

$$a_n = \frac{2n^2 + 3n}{\sqrt{5 + n^5}} \qquad b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}} \cdot \frac{n^{1/2}}{2} = \lim_{n\to\infty} \frac{2n^{5/2} + 3n^{3/2}}{2\sqrt{5 + n^5}}$$

$$= \lim_{n\to\infty} \frac{2 + \frac{3}{n}}{2\sqrt{\frac{5}{n^5} + 1}} = \frac{2 + 0}{2\sqrt{0 + 1}} = 1$$

Since $\sum b_n = 2\sum 1/n^{1/2}$ is divergent $\left(p\text{-series with } p = \frac{1}{2} < 1\right)$, the given series diverges by the Limit Comparison Test. ■

The Comparison Tests

The Limit Comparison Test

Theorem **The Limit Comparison Test**

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are infinite series with **positive** terms. Let $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.

1. • If L is a positive *number*, then (THINK: a_n is close to $\underbrace{L}_{\text{a positive number}} b_n$ for large n)

   $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.

2. • If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ **converges**, then (THINK: a_n is much smaller than b_n for large n)

   $\sum_{n=1}^{\infty} a_n$ also converges (ref: Sec 11.4 Exercise 40 on page 772).

3. • If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ **diverges**, then (THINK: a_n is much bigger than b_n for large n)

   $\sum_{n=1}^{\infty} a_n$ also diverges (ref: Sec 11.4 Exercise 41 on page 772).

Example:

Use the Limit Comparison Test to determine whether $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^3}$ converges

Let $a_n = \frac{\ln n}{n^3}$

Brainstorm: * Try $b_n = \frac{1}{n^3}$

But $\frac{a_n}{b_n} = \frac{\ln n}{n^3} \cdot \frac{n^3}{1} = \ln n \to \infty$ as $n \to \infty$ (can't apply LCT with this b_n since $\sum \frac{1}{n^3}$ converges)

Answer

* Try $b_n = \frac{1}{n^2}$

$$\frac{a_n}{b_n} = \frac{\ln n}{n^3} \cdot \frac{n^2}{1} = \frac{\ln n}{n}$$

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{1} = 0$$

(we can apply LCT 2 with this $b_n = \frac{1}{n^2}$ since $\sum \frac{1}{n^2}$ converges)

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges,

$\sum_{n=1}^{\infty} a_n$ also converges by Limit Comparison Test 2

The Comparison Tests

The Limit Comparison Test

Theorem **The Limit Comparison Test**

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are infinite series with **positive** terms. Let $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.

1. If L is a positive *number*, then (THINK: a_n is close to $\underbrace{L}_{\text{a positive number}} b_n$ for large n)

   $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.

2. If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ **converges**, then (THINK: a_n is much smaller than b_n for large n)

   $\sum_{n=1}^{\infty} a_n$ also converges (ref: Sec 11.4 Exercise 40 on page 772).

3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ **diverges**, then (THINK: a_n is much bigger than b_n for large n)

   $\sum_{n=1}^{\infty} a_n$ also diverges (ref: Sec 11.4 Exercise 41 on page 772).

Example:

Use the Limit Comparison Test to determine whether $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{n^2-4n+2}$ converges

Brainstorm: Try $b_n = \frac{\sqrt{n^3}}{n^2} = \frac{n^{\frac{3}{2}}}{n^2} = \frac{1}{n^{2-\frac{3}{2}}} = \frac{1}{n^{\frac{1}{2}}}$

or try $b_n = \frac{1}{n}$ — because I want to demonstrate part ③ of the Limit Comparison Test

Answer

Let $a_n = \frac{\sqrt{n^3+1}}{8n^2-4n+2}$ Let $b_n = \frac{1}{n}$

$$\frac{a_n}{b_n} = \frac{\sqrt{n^3+1}}{8n^2-4n+2} \cdot n = \frac{n\sqrt{n^3+1}}{8n^2-4n+2}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n\sqrt{n^3}}{8n^2} = \lim_{n\to\infty} \frac{n\, n^{\frac{3}{2}}}{8n^2} = \lim_{n\to\infty} \frac{n^{\frac{5}{2}}}{8n^2} = \lim_{n\to\infty} \frac{n^{\frac{1}{2}}}{8} = \infty$$

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series,

$\sum_{n=1}^{\infty} a_n$ is also divergent by Limit Comparison Test ③