Infinite Series

Sec 9.2

If we add the terms of a sequence $\{a_k\}_{k=1}^{n}$, we get an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n \qquad a_1+a_2+a_3 \qquad a_2+a_3+a_4+a_5$$

which is called a (finite) $\underbrace{\text{series}}_{\text{means "sum"}}$ and is also denoted by

$$\sum_{k=1}^{n} a_k. \qquad \sum_{k=1}^{3} a_k \qquad \sum_{k=2}^{5} a_k$$

specify starting index

Does it make sense to talk about the sum of infinitely many terms? Consider the **partial sums**

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3,$$

and, in general,

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k.$$

"First, take the sum of the first n terms"

If the sequence $\{S_n\}_{n=1}^{\infty} = \{S_1, S_2, S_3, \dots\}$ of partial sums has limit L, then we say that the infinite series **converges** to L and we write

$$\lim_{n\to\infty} S_n = L \quad \xleftrightarrow{\text{same meaning}} \quad \sum_{k=1}^{\infty} a_k = L$$

If the sequence $\{S_n\}_{n=1}^{n}$ of partial sums diverges, then we say that the infinite series **diverges**.

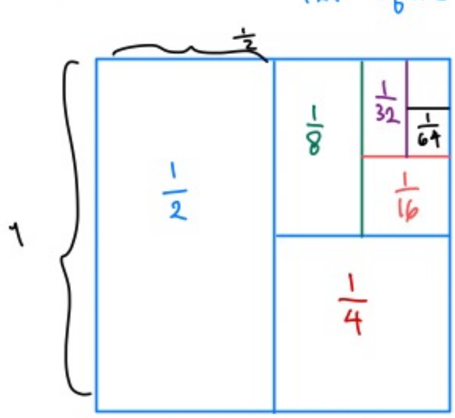
Summary(Notation)

• A **sequence** converges? $\lim_{k\to\infty} a_k = \underbrace{L}_{\text{(a number)}}$, $\{a_k\}$ converges	A **sequence** diverges? $\lim_{k\to\infty} a_k$ doesn't exist, $\{a_k\}$ diverges
• A **series** converges? $\lim_{n\to\infty} S_n = L$, $\sum_{k=1}^{\infty} a_k = \underbrace{L}_{\text{a number}}$, $\sum_{k=1}^{\infty} a_k$ converges	$\lim_{n\to\infty} S_n$ doesn't exist, A **series** diverges? $\sum_{k=1}^{\infty} a_k$ diverges

An important family of infinite series is the geometric series.

Visual example

1x1 square



$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \dots$

Area of 1x1 square is 1

So it seems like

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$

converges to 1

- A **geometric sequence** has the property that each term is obtained by multiplying the previous term by a fixed constant, called the **ratio**, e.g. $\underbrace{\{5, 10, 20, 40, 80, 160, \ldots\}}_{\text{ratio}=2}$.
- Given a geometric sequence $\{a_k\}_{k=1}^{\infty}$, if the ratio is r, then the k-th term can be expressed as $a_k = \underbrace{a_1}_{\text{first term}} r^{k-1}$, e.g. $a_k = 5 \; 2^{k-1}$ for $k = 1, 2, 3, \ldots$
- When $-1 < r \leq 1$, the sequence converges.

Geometric Series

Partial Sum of Geometric Series (Textbook Example 2)

Given a geometric sequence $\{a_k\}_{k=1}^{\infty}$, if the ratio is r, then the sum of the first n terms

$$S_n \overset{def}{=} a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-2} + a_1 r^{n-1}$$

e.g. $S_4 = a_1 + a_1 r + a_1 r^2 + a_1 r^3$

is (see below) .

Why?

$$\begin{array}{rccccccccc}
S_n = & a_1 + & a_1 r + & a_1 r^2 + & a_1 r^3 + & \ldots + & a_1 r^{n-2} + & a_1 r^{n-1} & \\
r\,S_n = & 0 & a_1 r + & a_1 r^2 + & a_1 r^3 + & \ldots + & a_1 r^{n-2} + & a_1 r^{n-1} + & a_1 r^n \\
\hline
S_n - r S_n = & a_1 + & 0 + & 0 + & \cdots & & & 0 & - a_1 r^n
\end{array} \; -$$

$$\text{Therefore,} \quad S_n - r\,S_n = a_1 - a_1 r^n,$$

$$S_n(1-r) = a_1(1-r^n)$$

$$\text{hence} \quad S_n = \frac{a_1(1-r^n)}{1-r} \quad \text{if } r \neq 1.$$

Furthermore, since

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{for } |r| < 1 \\ \text{DNE} & \text{for } r = -1 \\ \text{DNE} & \text{for } |r| > 1 \\ 1 & \text{for } r = 1 \end{cases}, \text{ we have } \lim_{n\to\infty} S_n = \lim_{n\to\infty} a_1 \frac{1-r^n}{1-r} = \begin{cases} a_1 \frac{1-0}{1-r} = a_1 \frac{1}{1-r} & \text{for } |r| < 1 \\ \text{DNE} & \text{for } r = -1 \\ \text{DNE} & \text{for } |r| > 1 \\ \text{DNE } (\infty) & \text{for } r = 1 \end{cases},$$

The partial sums $\{S_n\}$ do not converge if $|r| \geq 1$

Theorem (Geometric Series)

Let r and a be real numbers.

If $|r| < 1$, then $\sum_{k=1}^{\infty} ar^{k-1}$ $= a \frac{1}{1-r}$.

If $|r| \geq 1$, then $\sum_{k=1}^{\infty} ar^{k-1}$ diverges.

Note: In general, it's very hard to compute sums of (convergent) series

- The **geometric sequence** converges if and only if $-1 < r \leq 1$. (including 1)
- The **geometric series** converges if and only if $-1 < r < 1$. (not including 1)
- **Sum of a convergent series may change if you change your starting index:**

$\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots =$ 1, the area of a 1×1 square. $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k =$

1st term

$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots =$ $1 + 1 = 2$, the area of a 1×2 rectangle.

first term

If $|r| < 1$, then $\sum_{k=1}^{\infty} ar^{k} =$ $ar + ar^2 + ar^3 + \dots = ar\left[1 + r + r^2 + \dots\right] = ar \frac{1}{1-r}$

by above Thm (Geometric Series)

Example: Evaluate the (geometric) series $\sum_{k=1}^{\infty} \frac{3^k}{4^{k+3}}$ or state that it diverges.

step a.) State the test you plan to use: Geometric Series Thm (above)

step b.) (i) Write out the first 4 (four) terms of $\sum_{k=1}^{\infty} \frac{3^k}{4^{k+3}}$.

$\frac{3^1}{4^4}$ $\frac{3^2}{4^5}$ $\frac{3^3}{4^6}$ $\frac{3^4}{4^7}$ Note: To go from $\frac{3^2}{4^5}$ to $\frac{3^3}{4^6}$, multiply by $\frac{3}{4}$

(ii) Write our the first 4 (four) terms of $\sum_{k=1}^{\infty} ar^{k-1}$.

a ar^1 ar^2 ar^3 Note: To go from ar^1 to ar^2, multiply by r

(iii) Compare terms to find an a and an r so that $\sum_{k=1}^{\infty} \frac{3^k}{4^{k+3}} = \sum_{k=1}^{\infty} ar^{k-1}$.

$a = \frac{3}{4^4}$ $r = \frac{3}{4}$

step c.) After finding the ratio r, determine whether this geometric series converges or not.

Since $r = \frac{3}{4}$ is in $(-1, 1)$, the series converges to $a \frac{1}{1-r} = \frac{3}{4^4}\left(\frac{1}{1-\frac{3}{4}}\right) = \frac{3}{4^3}$

Telescoping Series

Find the sum of the "telescoping" series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$

Sol:

by partial fraction decomposition

Write $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \frac{1}{k} - \frac{1}{k+1}$$

$$S_4 = \sum_{k=1}^{4} \frac{1}{k} - \frac{1}{k+1} = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{5}\right)$$

$$= 1 - \frac{1}{5}$$

In general,

$$S_n = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1} = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cancel{} \dots \cancel{} + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} 1 - \frac{1}{n+1} = 1 - \lim_{n \to \infty} \frac{1}{n+1} = 1 - 0 = 1$$

So $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$

The series converges to 1.

Telescoping Series

Evaluate the series $\displaystyle\sum_{k=1}^{\infty} \ln\left(\frac{k}{k+4}\right)$ or state that it diverges.

Sol:

step a.) Find a formula for the k-th term of the sequence of **partial sums** $\{S_n\}$

$$
\begin{aligned}
S_n &= \sum_{k=1}^{n} \ln(k) - \ln(k+4) \\
&= \underbrace{\ln(1)}_{0} - \ln(\overset{1+4}{5}) + \ln(2) - \ln(\overset{2+4}{6}) + \ln(3) - \ln(\overset{3+4}{7}) + \ln(4) - \ln(\overset{4+4}{8}) \\
&\quad + \ln(5) - \ln(\underset{5+4}{9}) + \ln(6) - \ln(\underset{6+4}{10}) + \ln(7) - \ln(\underset{7+4}{11}) + \ln(8) - \ln(\underset{8+4}{12}) \\
&\quad + \cdots \\
-\ln(n-1) &\quad + \ln(\overset{k=n-4}{n-4}) - \ln(n) + \ln(\overset{k=n-3}{n-3}) - \ln(n+1) + \ln(\overset{k=n-2}{n-2}) - \ln(n+2) \\
&\quad + \ln(\underset{k=n-1}{n-1}) - \ln(n+3) + \ln(\underset{k=n}{n}) - \ln(n+4)
\end{aligned}
$$

$$= \underbrace{\ln(2) + \ln(3) + \ln(4)}_{\ln(2\cdot 3\cdot 4)} - \ln(n+1) - \ln(n+2) - \ln(n+3) - \ln(n+4)$$

step b.) Evaluate $\lim_{n\to\infty} S_n$ to obtain the sum of the series, or state that the series diverges.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \underbrace{\ln(24)}_{\text{a number}} - \underbrace{\ln(n+1) - \ln(n+2) - \ln(n+3) - \ln(n+4)}_{\text{go to } -\infty \text{ as } n\to\infty} = -\infty$$

∞

Note:

In general, $\displaystyle\sum_{k=1}^{n} f(k) - f(k+4) = f(1) + f(2) + f(3) + f(4)$
$\qquad - f(n+1) - f(n+2) - f(n-3) - f(n-4)$

Thm

If the series $\sum_{k=1}^{\infty} a_k$ is convergent, then $\lim_{k\to\infty} a_k$ $= 0$.

What does this theorem say? Recall that to any series $\sum a_n$ we associate two sequences:

- the sequence $\{a_k\}$ of its **terms**, and
- the sequence $\{S_n\}$ of its **partial sums**.

The theorem says that if $\sum_{k=1}^{n} a_k$ converges to a number S, then

$$\lim_{n\to\infty} S_n = S \qquad \text{and} \qquad \lim_{k\to\infty} a_k = 0$$

Caution: If the series $\sum_{k=1}^{\infty} a_k$ is divergent, then $\lim_{k\to\infty} a_k$ we cannot say – it depends

Vocab What is the contrapositive of a statement? Statement: IF P THEN Q

Contrapositive of this statement is "IF (NOT Q) THEN (NOT P)"

The contrapositive is equivalent to the original statement

E.g. Statement: IF UML has snow day, THEN it is snowing

Contrapositive: IF it's not snowing, THEN UML has no snow day.

Note: If it's snowing, we don't know whether there will be a snow day.

nth-Term Test for Divergence:

If $\lim_{k\to\infty} a_k \neq 0$ OR if $\lim_{k\to\infty} a_k$ doesn't exist, then the series $\sum_{k=1}^{\infty} a_k$ is NOT convergent.

This statement is the contrapositive of the Thm at the top of this page

Caution: If $\lim_{k\to\infty} a_k = 0$, then **the test is inconclusive**. We cannot use this test to determine convergence/divergence of $\sum a_k$.

Example: Use the Test for Divergence to determine whether the series $\sum_{k=1}^{\infty} \frac{k}{2k+1}$ diverges, or state that the Test for Divergence is inconclusive.

First step: $\lim_{k\to\infty} \frac{k}{2k+1} = \lim_{k\to\infty} \frac{\left(\frac{k}{k}\right)}{\left(2\frac{k}{k}\right)+\frac{1}{k}} = \lim_{k\to\infty} \frac{1}{2+\frac{1}{k}} = \frac{1}{2}$

Second step: The Test for Divergence is conclusive/ inconclusive

Since $\lim_{k\to\infty} \frac{k}{2k+1} \neq 0$, the series $\sum \frac{k}{2k+1}$ is divergent

Example: Use the Test for Divergence to determine whether the series $\sum_{k=1}^{\infty} \frac{k}{k^2+1}$ diverges, or state that the Test for Divergence is inconclusive.

First step: $\lim_{k\to\infty} \frac{k}{k^2+1} \underset{\text{L'H } \frac{\infty}{\infty}}{=} \lim_{k\to\infty} \frac{1}{2k} = 0$

Second step: The Test for Divergence is conclusive/inconclusive Test for Divergence doesn't help.

Properties of Convergent Series

Suppose c is a number. If $\sum a_k$ and $\sum b_k$ are convergent series, ...

- then the series $\sum c\ a_k$ also **converges** and $\sum c\ a_k =$ $c \sum a_k$
- then the series $\sum a_k + \sum b_k$ also **converges** and $\sum a_k + b_k =$ $\sum a_k + \sum b_k$

Example: Evaluate $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n} \right)$ or state that it diverges.

step a.) First compute $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ to get 1.

Earlier Ex

step b.) Next, compute $\sum_{n=1}^{\infty} \frac{1}{2^n}$ to get 1.

Earlier Ex

step c.) $\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n} \right) = 3 \left(\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \right) + \left(\sum_{n=1}^{\infty} \frac{1}{2^n} \right) = 3 \cdot 1 + 1.$

Apply Properties

Repeating Decimals

Example:

Write $0.9\overline{34} = 0.93434343434\ldots$ as a geometric series and express its value as a fraction.

a.) Can you write $.9\overline{34} = .93434343434\ldots$ as geometric series?

$$0.9 + \frac{34}{10^3} + \frac{34}{10^5} + \frac{34}{10^7} + \cdots$$

b.) If $0.0\overline{34} = 0.03434343434\cdots = \sum_{k=1}^{\infty} ar^{k-1}$, what is a?

First term $a = \frac{34}{10^3}$

c.) If $0.0\overline{34} = 0.03434343434\cdots = \sum_{k=1}^{\infty} ar^{k-1}$, what is r?

$r = \frac{1}{100}$ The constant we multiply by to get to the next term

d.) Is $|r| < 1$?

yes $\left|\frac{1}{10^2}\right| < 1$

e.) Use the geometric series found in the previous parts to convert $0.9\overline{34} = 0.93434343434\ldots$ into a fraction.

$$0.9\overline{34} = 0.9 + \frac{34}{10^3} + \frac{34}{10^5} + \frac{34}{10^7} + \cdots$$
$$= 0.9 + \frac{34}{10^3}\left[1 + \frac{1}{100} + \frac{1}{100^2} + \frac{1}{100^3} + \cdots\right]$$
$$= 0.9 + \frac{34}{10^3} \cdot \frac{1}{1 - \frac{1}{100}} = 0.9 + \frac{34}{10^3} \cdot \frac{1}{\left(\frac{99}{100}\right)}$$

f.) Perform a reality check, ~~for example~~, verify that your fraction is between $\frac{9}{10}$ and 1.

$$= 0.9 + \frac{34}{990} = \frac{9}{10} + \frac{34}{990} = \_\_\_\_\_.$$

Repeating Decimals

Example:

Write as a geometric series then express its value as a fraction.

- $0.\overline{38}$ $\quad$ $1.2\overline{38}$
- $0.2\overline{74}$ $\quad$ $1.2\overline{74}$

Solutions below

Solutions

1.

$$\begin{aligned} 0.\overline{38} &= 0.383838\cdots \\ &= \frac{38}{100} + \frac{38}{100^2} + \frac{38}{100^3} + \cdots \\ &= \frac{\frac{38}{100}}{1 - \frac{1}{100}} \\ &= \frac{38}{99} \end{aligned}$$

2.

$$\begin{aligned} 1.\overline{38} &= 1 + 0.\overline{38} \\ &= 1 + \frac{38}{99} \\ &= \frac{99 + 38}{99} \\ &= \frac{(100 - 1) + 38}{99} \\ &= \frac{138 - 1}{99} \end{aligned}$$

3.

$$\begin{aligned} 0.2\overline{74} &= 0.2 + 0.0747474\cdots \\ &= \frac{2}{10} + \frac{74}{1000} + \frac{74}{1000 \times 100} + \frac{74}{1000 \times 100^2} + \cdots \\ &= \frac{2}{10} + \frac{\frac{74}{1000}}{1 - \frac{1}{100}} \\ &= \frac{2}{10} + \frac{74}{990} \\ &= \frac{2 \times 99 + 74}{990} \\ &= \frac{2 \times (100 - 1) + 74}{990} \\ &= \frac{274 - 2}{990} \end{aligned}$$

4.

$$\begin{aligned} 1.2\overline{74} &= 1 + 0.2\overline{74} \\ &= 1 + \frac{274 - 2}{990} \\ &= \frac{990 + 274 - 2}{990} \\ &= \frac{(1000 - 10) + 274 - 2}{990} \\ &= \frac{1274 - 12}{990} \end{aligned}$$