9.10 Applications of Taylor Series

$$(1+x)^0 = 1$$
$$(1+x)^1 = 1+x$$
$$(1+x)^2 = 1+2x+x^2$$
$$(1+x)^3 = 1+3x+3x^2+x^3$$
$$(1+x)^4 = 1+4x+6x^2+4x^3+x^4$$
$$(1+x)^5 = 1+5x+10x^2+10x^3+5x^4+x^5$$

Pattern

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Def: The binomial coefficients are

$$\binom{m}{0} := 1, \quad \binom{m}{1} := m, \quad \binom{m}{2} := \frac{\overbrace{m(m-1)}^{2 \text{ factors}}}{2!},$$

$$\binom{m}{k} := \frac{\overbrace{m(m-1)(m-2)\cdots(m-k+1)}^{k \text{ factors}}}{k!} \quad \text{for } k \geq 3$$

Read "m choose k" because this is the number of ways to choose k objects out of m if m is a positive number.

Ex: I have four students Caroline, John, Lindsey, Matthew. If I want to randomly choose two of them to present, there are $\binom{4}{2} = \frac{4(3)}{2!} = 6$ possibilities:

3 is 4-2+1

CJ, JL, LM
CL, JM,
CM

The binomial Series (Theorem)

The Taylor series for $(1+x)^m$ is $\sum_{k=0}^{\infty} \binom{m}{k} x^k$

$$\boxed{(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k \quad \text{for} \quad -1 < x < 1}$$

Ex: If $m = 3$,

$\binom{3}{0} = 1$, $\binom{3}{1} = m = 3$, $\binom{3}{2} = \frac{3\overbrace{(2)}^{m-k+1\,=\,3-2+1}}{2!} = 3$, $\binom{3}{3} = \frac{3(2)\overbrace{(1)}^{3-3+1}}{3!} = 1$,

$\binom{3}{4} = 0$, $\binom{3}{5} = 0$, $\binom{3}{k} = 0$ if $k > 3$.

In general, if m is a positive integer, $\binom{m}{k} = 0$ if $k > m$.

So $(1+x)^3 = \binom{3}{0}x^0 + \binom{3}{1}x + \binom{3}{2}x^2 + \binom{3}{3}x^3$

$= 1 + 3x + 3x^2 + x^3$

The Taylor series is finite (a polynomial)

Ex: If $m = -1$,

$\binom{-1}{0} = 1$, $\binom{-1}{1} = m = -1$, $\binom{-1}{2} = \frac{-1(-2)}{2!} = 1$,

$\binom{-1}{3} = \frac{-1(-2)\overbrace{(-3)}^{-1-3+1}}{3!} = (-1)\frac{3!}{3!} = -1$

Note: We know from Sec 9.7 that $(1+x)^{-1} = \frac{1}{1-(-x)} = \sum_{k=0}^{\infty} (-1)^k x^k$ for $|x| < 1$

for $k = 0, 1, 2, \ldots$

In general, $\binom{-1}{k} = \frac{\overbrace{(-1)(-2)(-3)\cdots(-1-k+1)}^{k \text{ products}}}{k!} = (-1)^k \frac{k!}{k!} = (-1)^k$

k

So $(1+x)^{-1} = \sum_{k=1}^{\infty} \binom{-1}{k} x^k = \sum_{k=1}^{\infty} (-1)^k x^k$ an infinite power series

Ex: Write the first four nonzero terms of the Taylor series for the function $(1+3x)^{\frac{1}{2}}$

Sol: $m=\frac{1}{2}$

$$\binom{m}{0}=1,\quad \binom{m}{1}=m=\frac{1}{2},\quad \binom{m}{2}=\frac{\overbrace{\frac{1}{2}\left(\frac{1}{2}-1\right)}^{\text{two factors because } k=2}}{2!}=\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}=-\frac{1}{2^3},$$

$$\binom{m}{3}=\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}=\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}=\frac{3}{2^3(3)2}=\frac{1}{2^4}$$

$$(1+3x)^{\frac{1}{2}}=\sum_{k=0}^{\infty}\binom{m}{k}(3x)^k \qquad \text{for } |3x|<1$$

$$=1(3x)^0+\frac{1}{2}(3x)^1-\frac{1}{2^3}(3x)^2+\frac{1}{2^4}(3x)^3+\dots$$

$$=\boxed{1+\frac{3}{2}x-\frac{9}{8}x^2+\frac{27}{16}x^3}+\dots$$

first four nonzero terms

Ex: Write the first four nonzero terms of the Taylor series for the function $(1+x^3)^{-\frac{1}{3}}$

Sol: $m=-\frac{1}{3}$

$$\binom{m}{0}=1,\quad \binom{m}{1}=m=-\frac{1}{3},\quad \binom{m}{2}=\frac{\overbrace{-\frac{1}{3}\left(-\frac{1}{3}-1\right)}^{2\text{ factors}}}{2!}=\frac{-\frac{1}{3}\left(-\frac{4}{3}\right)}{2}=\frac{4}{9}\cdot\frac{1}{2}=\frac{2}{9},$$

$$\binom{m}{3}=\frac{-\frac{1}{3}\left(-\frac{1}{3}-1\right)\left(-\frac{1}{3}-2\right)}{3!}=\frac{-\frac{1}{3}\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{3(2)}=-\frac{4(7)}{3^3}\,\frac{1}{3(2)}=\frac{-14}{81}$$

So $$(1+x^3)^{-\frac{1}{3}}=\sum_{k=0}^{\infty}\binom{-\frac{1}{3}}{k}(x^3)^k=1-\frac{1}{3}(x^3)+\frac{2}{9}(x^3)^2-\frac{14}{81}(x^3)^3+\dots$$

$$=\boxed{1-\frac{x^3}{3}+\frac{2}{9}x^6-\frac{14}{81}x^9}+\dots$$

first four nonzero terms

TABLE 9.1 Frequently Used Taylor Series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \qquad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \qquad -1 < x \le 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad |x| \le 1$$

Evaluating nonelementary integrals (review of Sec 9.7)

Ex:

Integrals like $\int \sin(x^2)\,dx$ arise in the study of the diffraction of light. It cannot be expressed as an elementary function, but it can be expressed as a power series.

$$\sin(x^2) = (x^2) - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \ldots = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

$$\int \sin(x^2)\,dx = C + \underbrace{\frac{x^3}{3}}_{n=0} - \underbrace{\frac{x^7}{7\cdot 3!}}_{n=1} + \underbrace{\frac{x^{11}}{11\cdot 5!}}_{n=2} - \underbrace{\frac{x^{15}}{15\cdot 7!}}_{n=3} = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)\cdot(2n+1)!}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \qquad |x| < \infty$$

Def:

$(i)^2 = -1$

$(i)^3 = i(i)^2 = -i$

$(i)^4 = i^2 i^2 = (-1)^2 = 1$

$(i)^5 = i(i^4) = i$

1) $$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \ldots$$

$$= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \ldots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \ldots$$

2) $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!},$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \ldots$$

3) $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!},$$

$$i\sin\theta = i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots\right)$$

$$= i\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} - \frac{i\theta^7}{7!} + \ldots$$

$$\boxed{\cos\theta + i\sin\theta = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \frac{\theta^8}{8!} - \ldots = e^{i\theta}}$$

For any real number θ,

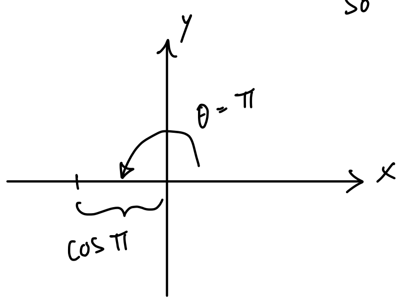
$$e^{i\theta} = \cos\theta + i\sin\theta \qquad (\text{called "Euler's identity"})$$

So, for any complex number $a+ib$, we have

$$e^{a+ib} = e^{a}(e^{ib}) = e^{a}(\cos b + i\sin b) = e^{a}\cos b + e^{a}\, i\sin b$$

For example, $e^{i\pi} = \cos\pi + i\sin\pi = -1 + i0$

So $\boxed{e^{i\pi} = -1}$



Remark:

This explanation $e^{i\theta} = \cos\theta + i\sin\theta$

relates power series (last part of Ch 9)

with polar coordinates (Ch 10) !