Sec 8.7 Improper Integrals (Type I)

An application

Work required to move a rocket from the surface of earth to another point t meters (from the center of earth) is

$$\int_R^t G\,m\,M\,\frac{1}{x^2}\,dx$$ in joule or newton-meter,

where G is a constant in $\frac{\text{newton meters}^2}{\text{kg}^2}$ (called gravitational constant),

m is the mass in kg of the rocket,

M is the mass in kg of earth,

R is the radius of earth.

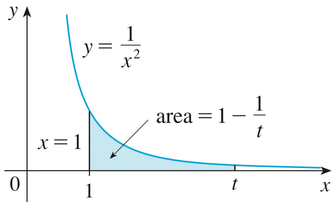
Evaluate:

$$GmM\int_R^t x^{-2}\,dx = GmM\left.\frac{x^{-1}}{-1}\right|_R^t = GmM\left(-\frac{1}{t}+\frac{1}{R}\right)$$

The work required to free the rocket from earth's gravity pull

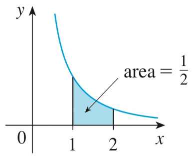
is $$\underbrace{\lim_{t\to\infty}\int_R^t \frac{GmM}{x^2}\,dx}_{} = \lim_{t\to\infty} GmM\left(-\frac{1}{t}+\frac{1}{R}\right) = GmM\left(\frac{1}{R}\right)$$

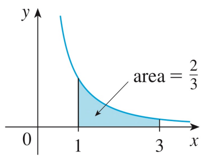
Vocab: this concept is called an improper integral, even when the limit does not exist

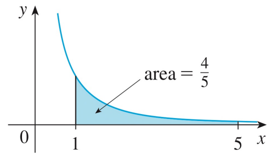


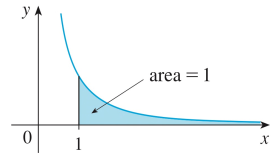
Ex 1

$$A(t) = \int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_1^t = 1 - \frac{1}{t}$$









$\int_1^2 \frac{1}{x^2}\,dx = 1 - \frac{1}{2}$ $\qquad \int_1^3 \frac{1}{x^2}\,dx = 1 - \frac{1}{3}$ $\qquad \int_1^4 \frac{1}{x^2}\,dx = 1 - \frac{1}{4}$ $\qquad \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} 1 - \frac{1}{t}$

$= 1 - 0$

New notation: $\displaystyle \int_1^\infty \frac{1}{x^2}\,dx \overset{\text{Def}}{=} \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx$ (if the right hand side exists)

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1 Definition of an Improper Integral of Type 1

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geqslant a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leqslant b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

Example: $\displaystyle \int_1^\infty \frac{1}{x^2}\,dx \overset{\text{def}}{=} \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx$

since the RHS is equal to a number

- We say $\int_1^\infty \frac{1}{x^2}\,dx$ is $\boxed{\text{CONVERGENT}}$ because $\boxed{\lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx \text{ equals a number.}}$

Ex 2

- $\displaystyle \int_t^{-1} \frac{1}{x}\,dx = \ln|x| \Big|_{x=t}^{x=-1} = \ln|-1| - \ln|t| = -\ln|t|$

So $\displaystyle \lim_{t\to-\infty} \int_t^{-1} \frac{1}{x}\,dx = \lim_{t\to-\infty} -\ln|t| = -\infty$ so $\displaystyle \int_{-\infty}^{-1} \frac{1}{x}\,dx$ is DIVERGENT

The Integral $\int_1^{\infty} \frac{dx}{x^p}$

The function $y = 1/x$ is the boundary between the convergent and divergent improper integrals with integrands of the form $y = 1/x^p$. As the next example shows, the improper integral converges if $p > 1$ and diverges if $p \leq 1$.

EXAMPLE 3 For what values of p does the integral $\int_1^{\infty} dx/x^p$ converge? When the integral does converge, what is its value?

Solution If $p \neq 1$, then

$$\int_1^b \frac{dx}{x^p} = \frac{x^{-p+1}}{-p+1}\bigg]_1^b = \frac{1}{1-p}(b^{-p+1} - 1) = \frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right).$$

Thus,

$$\begin{aligned}\int_1^{\infty} \frac{dx}{x^p} &= \lim_{b\to\infty} \int_1^b \frac{dx}{x^p} \\ &= \lim_{b\to\infty}\left[\frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right)\right] = \begin{cases} \dfrac{1}{p-1}, & p > 1 \\ \infty, & p < 1 \end{cases}\end{aligned}$$

because

$$\lim_{b\to\infty} \frac{1}{b^{p-1}} = \begin{cases} 0, & p > 1 \\ \infty, & p < 1. \end{cases}$$

Therefore, the integral converges to the value $1/(p-1)$ if $p > 1$, and it diverges if $p < 1$.

If $p = 1$, the integral also diverges:

$$\begin{aligned}\int_1^{\infty} \frac{dx}{x^p} &= \int_1^{\infty} \frac{dx}{x} \\ &= \lim_{b\to\infty} \int_1^b \frac{dx}{x} \\ &= \lim_{b\to\infty} \Big[\ln|x|\Big]_1^b \\ &= \lim_{b\to\infty} (\ln b - \ln 1) = \infty. \qquad \blacksquare\end{aligned}$$

1 Definition of an Improper Integral of Type 1

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^{\infty} f(x)\,dx \overset{\text{def}}{=} \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$$

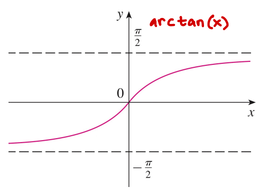
any number a will result in the same

Warning • If $\int_a^\infty f(x)\,dx$ or $\int_{-\infty}^a f(x)\,dx$ is divergent, then we say $\int_{-\infty}^\infty f(x)\,dx$ is divergent.

• Both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ must be convergent for $\int_{-\infty}^\infty f(x)$ to be convergent.

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$ **Ex 3**

$$\int_0^t \frac{1}{1+x^2}\,dx = \arctan(x)\Big|_0^t = \arctan(t) - \underbrace{\arctan(0)}_{0}$$

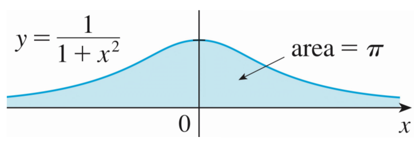


$$\lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \arctan(t) = \frac{\pi}{2}$$

$$\int_t^0 \frac{1}{1+x^2}\,dx = \arctan(x)\Big|_t^0 = \underbrace{\arctan(0)}_{0} - \arctan(t)$$

$$\lim_{t\to-\infty} \int_t^0 \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} -\arctan(t) = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

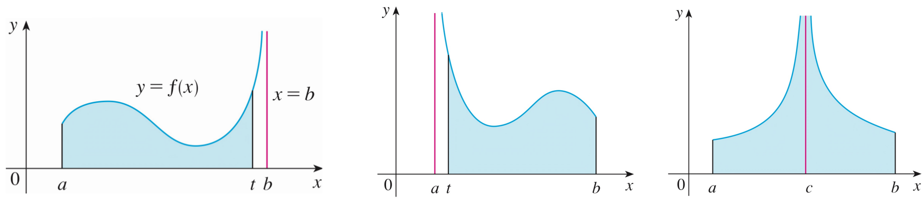
So $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_0^{\infty} \frac{1}{1+x^2}\,dx + \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2}$



Sec 8.7 Improper Integrals (Type II)

The symbol $\int_a^b f(x)\,dx$ means an improper integral

if $f(x)$ has an infinite discontinuity in $[a,b]$



3 Definition of an Improper Integral of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

From now on, when you see $\int_a^b f(x)\,dx$, you need to first determine whether it's improper integral by checking for infinite discontinuity of f in $[a,b]$.

Find $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$ **Ex 4**

Because $\displaystyle\lim_{x\to 2^+}\frac{1}{\sqrt{x-2}} = \infty$, the function $\dfrac{1}{\sqrt{x-2}}$ has an infinite discontinuity in $[2,5]$, so the symbol $\displaystyle\int_2^5\frac{1}{\sqrt{x-2}}dx$ means improper integral.

$$\int_t^5 \frac{1}{\sqrt{x-2}}dx = \int_t^5 (x-2)^{-\frac{1}{2}}\,dx = \left.\frac{(x-2)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}\right|_t^5 = 2\left(\sqrt{5-2}-\sqrt{t-2}\right)$$

$$\lim_{t\to 2^+}\int_t^5 \frac{1}{\sqrt{x-2}} = \lim_{t\to 2^+} 2\left(\sqrt{3}-\sqrt{t-2}\right) = 2\sqrt{3}$$

So $\displaystyle\int_2^5\frac{1}{\sqrt{x-2}} = 2\sqrt{3}$, so it is CONVERGENT.

Example

$\displaystyle\int_2^5\frac{1}{\sqrt{x-2}}dx$ is an improper integral (new concept!)

$\displaystyle\int_4^7\frac{1}{\sqrt{x-2}}dx$ is a usual definite integral (Ch 5 concept)

3 Definition of an Improper Integral of Type 2

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

If one of $\int_a^c f(x)\,dx$ or $\int_c^b f(x)\,dx$ is divergent, then $\int_a^b f(x)\,dx$ is divergent.

Evaluate $\displaystyle\int_0^3 \frac{dx}{x-1}$ if possible **Ex 5**

WARNING WRONG COMPUTATION WARNING

$$\int_0^3 \frac{1}{x-1}\,dx = \ln|x-1|\Big|_0^3 = \ln(2) - \underbrace{\ln(1)}_{0} = \ln(2)$$

Improper integrals are not integrals

Because $\displaystyle\lim_{x\to 1^+} \frac{1}{x-1} = \boxed{\infty}$ and $\displaystyle\lim_{x\to 1^-} \frac{1}{x-1} = \boxed{-\infty}$

the function $\dfrac{1}{x-1}$ has an infinite discontinuity at $x=1$,

so the symbol $\displaystyle\int_0^3 \frac{1}{x-1}\,dx$ means improper integral. (new concept! Sec 7.8)

Example: $\displaystyle\int_4^6 \frac{1}{x-1}\,dx$ is a usual definite integral (from Chapter 5)

Need to check the convergence / divergence of

$\displaystyle\int_1^3 \frac{1}{x-1}\,dx$ and $\displaystyle\int_0^1 \frac{1}{x-1}\,dx$ separately.

$$\int_0^t \frac{1}{x-1}\,dx = \ln|x-1|\Big|_0^t = \ln|t-1| - \ln|0-1| = \ln|t-1| - \underbrace{\ln(1)}_{0}$$

$$\lim_{t\to 1^-} \int_0^t \frac{1}{x-1}\,dx = \lim_{t\to 1^-} \ln|t-1| = -\infty$$

So $\displaystyle\int_0^1 \frac{1}{x-1}\,dx = -\infty$, and we call $\displaystyle\int_0^1 \frac{1}{x-1}\,dx$ divergent.

Without even checking $\displaystyle\int_1^3 \frac{1}{x-1}\,dx$, we can say $\displaystyle\int_0^3 \frac{1}{x-1}\,dx$ is divergent.

Thm (Direct Comparison Test for Improper Integrals)

① IF $0 \le f(x) \le g(x)$ for $x \ge a$ and $\int_a^\infty \underbrace{g(x)}_{\text{"bigger"}}\,dx$ is convergent,

THEN $\int_a^\infty \underbrace{f(x)}_{\text{"smaller"}}\,dx$ is convergent.

② IF $0 \le f(x) \le g(x)$ for $x \ge a$ and $\int_a^\infty \underbrace{f(x)}_{\text{"smaller"}}\,dx = \infty$,

(Vocab: We say $\int_a^\infty f(x)\,dx$ is divergent)

THEN $\int_a^\infty \underbrace{g(x)}_{\text{"bigger"}}\,dx = \infty$.

vocab: We say $\int_a^\infty g(x)\,dx$ is divergent

Determine whether $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ is convergent. **Ex 6**

(Note: $\int \frac{1+e^{-x}}{x}\,dx$ cannot be expressed as an elementary function, so we cannot evaluate it using Chapter 8 methods)

Answer

- $0 \le \frac{1}{x} \le \frac{1+e^{-x}}{x}$ for $x \ge 1$

- $\displaystyle\int_1^\infty \frac{1}{x}\,dx \overset{\text{Def}}{=} \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln|x|\Big|_1^t = \lim_{t\to\infty} \ln|t| - \ln(1) = \infty$

- By the Direct Comparison Thm for Improper Integrals,

$$\int_1^\infty \frac{1+e^{-x}}{x}\,dx = \infty$$

so it is not convergent.

(Another word for "not convergent" is divergent.)

More examples of using the Direct Comparison Test from the textbook

THEOREM 2—Direct Comparison Test
Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

1. If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then $\displaystyle\int_a^\infty f(x)\,dx$ also converges.
2. If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then $\displaystyle\int_a^\infty g(x)\,dx$ also diverges.

EXAMPLE 7 These examples illustrate how we use Theorem 2.

(a) $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converges because

$$0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2} \quad \text{on} \quad [1, \infty) \quad \text{and} \quad \int_1^\infty \frac{1}{x^2}\,dx \quad \text{converges.}$$ Example 3

(b) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^2 - 0.1}}\,dx$ diverges because

$$\frac{1}{\sqrt{x^2 - 0.1}} \geq \frac{1}{x} \quad \text{on} \quad [1, \infty) \quad \text{and} \quad \int_1^\infty \frac{1}{x}\,dx \quad \text{diverges.}$$ Example 3

(c) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}}\,dx$ converges because

$$0 \leq \frac{\cos x}{\sqrt{x}} \leq \frac{1}{\sqrt{x}} \quad \text{on} \quad \left[0, \frac{\pi}{2}\right],$$ $0 \leq \cos x \leq 1$ on $\left[0, \frac{\pi}{2}\right]$

and

$$\int_0^{\pi/2} \frac{dx}{\sqrt{x}} = \lim_{a \to 0^+} \int_a^{\pi/2} \frac{dx}{\sqrt{x}}$$

$$= \lim_{a \to 0^+} \sqrt{4x}\,\Big]_a^{\pi/2}$$ $2\sqrt{x} = \sqrt{4x}$

$$= \lim_{a \to 0^+} \left(\sqrt{2\pi} - \sqrt{4a}\right) = \sqrt{2\pi} \quad \text{converges.}$$

Thm(Limit Comparison Test for Improper Integrals)

THEOREM 3—Limit Comparison Test
If the positive functions f and g are continuous on $[a, \infty)$, and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

either *both converge* or *both diverge*.

Although the improper integrals of two functions from a to ∞ may both converge, this does not mean that their integrals necessarily have the same value, as the next example shows.

EXAMPLE 8 Show that

$$\int_1^\infty \frac{dx}{1 + x^2}$$

converges by comparison with $\int_1^\infty (1/x^2)\,dx$. Find and compare the two integral values.

Solution The functions $f(x) = 1/x^2$ and $g(x) = 1/(1 + x^2)$ are positive and continuous on $[1, \infty)$. Also,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1/x^2}{1/(1 + x^2)} = \lim_{x\to\infty} \frac{1 + x^2}{x^2}$$
$$= \lim_{x\to\infty} \left(\frac{1}{x^2} + 1\right) = 0 + 1 = 1,$$

which is a positive finite limit (Figure 8.20). Therefore, $\int_1^\infty \frac{dx}{1 + x^2}$ converges because $\int_1^\infty \frac{dx}{x^2}$ converges.

The integrals converge to different values, however:

$$\int_1^\infty \frac{dx}{x^2} = \frac{1}{2 - 1} = 1 \qquad \text{Example 3}$$

and

$$\int_1^\infty \frac{dx}{1 + x^2} = \lim_{b\to\infty} \int_1^b \frac{dx}{1 + x^2} = \lim_{b\to\infty} \left[\tan^{-1} b - \tan^{-1} 1\right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

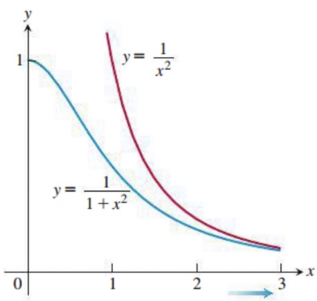


FIGURE 8.20 The functions in Example 8.

EXAMPLE 9 Investigate the convergence of $\int_1^\infty \frac{1 - e^{-x}}{x}\,dx$.

Solution The integrand suggests a comparison of $f(x) = (1 - e^{-x})/x$ with $g(x) = 1/x$. However, we cannot use the Direct Comparison Test because $f(x) \le g(x)$ and the integral of $g(x)$ *diverges*. On the other hand, using the Limit Comparison Test, we find that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \left(\frac{1 - e^{-x}}{x}\right)\left(\frac{x}{1}\right) = \lim_{x\to\infty} (1 - e^{-x}) = 1,$$

which is a positive finite limit. Therefore, $\int_1^\infty \frac{1 - e^{-x}}{x}\,dx$ diverges because $\int_1^\infty \frac{dx}{x}$ diverges. Approximations to the improper integral are given in Table 8.5. Note that the values do not appear to approach any fixed limiting value as $b \to \infty$.