Sec 8.6 Numerical Integration

Def: Elementary functions

An elementary function is a function which can be obtained from

power functions (like $x^{\frac{1}{5}}$),
exponential functions (like e^x),
logarithmic functions (like $\ln$),
trig functions (like $\sec(x)$),
inverse trig functions (like $\operatorname{arcsec}(x)$)

by addition/subtraction/multiplication/division/composition

Example

$$f(x) = \sqrt{x^3+1} + \frac{1}{\ln x} - e^{(x^2)}$$

- If f is an elementary function, then f' is also an elementary function

but $\int f(x)\,dx$ may not be an elementary function.

— In fact, the majority of elementary functions do not have elementary antiderivatives (and we cannot compute them using u-substitution or Sec 8.1-8.4 methods)

These integrals cannot be expressed as elementary functions:

$$\int e^{x^2}\,dx \qquad \int \frac{e^x}{x}\,dx \qquad \int \sin(x^2)\,dx \qquad \int \cos(e^x)\,dx \qquad \int \ln(\ln x)\,dx$$

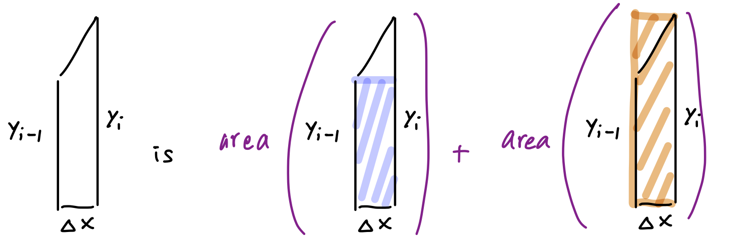
$$\int e^{(e^x)}\,dx \qquad \int \sqrt{x^3+1}\,dx \qquad \int \frac{1}{\ln x}\,dx \qquad \int \frac{\sin x}{x}\,dx$$

We'll be able to express them in Sec 9.10

Trapezoid Rule for definite integrals

Idea: Instead of rectangles, approximate using trapezoids.

Area of a trapezoid



$$= \frac{Y_{i-1}\,\Delta x + Y_i\,\Delta x}{2} = \frac{\Delta x}{2}\left(Y_{i-1} + Y_i\right)$$

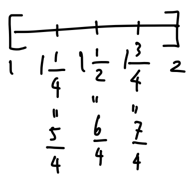
Ex 1:

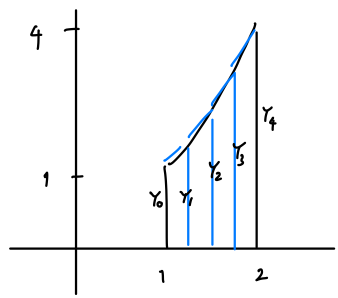
a) Use the Trapezoidal Rule with $n = 4$ steps to estimate $\int_1^2 x^2\,dx$.

b) Compare the estimate with the exact value.

Sol: a) Partition the interval $[1, 2]$ into 4 subintervals:

$$\Delta x = \frac{2-1}{4} = \frac{1}{4}$$





Approximation is $T = \frac{\Delta x}{2}(Y_0 + Y_1) + \frac{\Delta x}{2}(Y_1 + Y_2) + \frac{\Delta x}{2}(Y_2 + Y_3) + \frac{\Delta x}{2}(Y_3 + Y_4)$

$$= \frac{\left(\frac{1}{4}\right)}{2}\left(Y_0 + 2Y_1 + 2Y_2 + 2Y_3 + Y_4\right)$$

$$= \frac{1}{8}\left(1^2 + 2\left(\frac{5}{4}\right)^2 + 2\left(\frac{6}{4}\right)^2 + 2\left(\frac{7}{4}\right)^2 + 2^2\right)$$

$$= \boxed{\frac{75}{32}} = 2.34375$$

b) Exact value is $\int_1^2 x^2\,dx = \frac{x^3}{3}\Big|_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$

Error: $2.34375 - \frac{7}{3} \approx 0.0104$. Percentage error: $\left(2.34375 - 7/3\right) / \left(\frac{7}{3}\right) \cdot 100\% \approx 0.446\%$.

Thm: Let M be any upper bound for the values of $|f''|$ on $[a,b]$.

Then the error E_T in the trapezoidal approximation of $\int_a^b f(x)\,dx$ using n steps satisfies

$$|E_T| \leq \frac{M\,(b-a)^3}{12\,n^2}$$

Ex 1: c) Find an upper bound for $|E_T|$ for the Trapezoidal rule approximation of $\int_1^2 x^2\,dx$ with $n = 4$ steps.

Sol: $f(x) = x^2$

$f'(x) = 2x$

$f''(x) = 2$

So $M = 2$

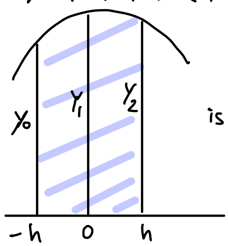
an upper bound for $|E_T|$

$$|E_T| \leq \boxed{\frac{2\,(2-1)^2}{12\,(4)^2} = \frac{1}{6}\,\frac{1}{16}} \approx 0.0104$$

Simpson's Rule

Approximate using parabolas (instead of straight-lines).

$Y = Ax^2 + Bx + C$



Area of is $A = \int_{-h}^{h} (Ax^2 + Bx + C)\,dx = \frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx \Big|_{-h}^{h}$

$$= \frac{Ah^3}{3} + \cancel{\frac{Bh^2}{2}} + Ch$$

$$- \frac{A(-h)^3}{3} - \cancel{\frac{B(-h)^2}{2}} - C(-h)$$

$$= \frac{2}{3}Ah^3 + 2Ch$$

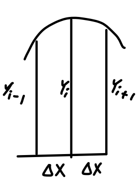
$$= \frac{h}{3}\left(2Ah^2 + 6C\right)$$

We know
$$Y_0 = A(-h)^2 + B(-h) + C = Ah^2 - Bh + C$$
$$Y_1 = A(0)^2 + B(0) + C = C$$
$$Y_2 = A(h)^2 + Bh + C$$

So
$$Y_0 + 4Y_1 + Y_2 = Ah^2 - Bh + C$$
$$+ 4C$$
$$Ah^2 + Bh + C$$
$$= 2Ah^2 + 6C$$

So area $A = \frac{h}{3}\left(2Ah^2 + 6C\right) = \frac{h}{3}\left(Y_0 + 4Y_1 + Y_2\right)$.

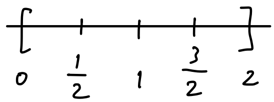
Note: For Simpson's rule we need an even number of subintervals because each shape requires two subintervals.



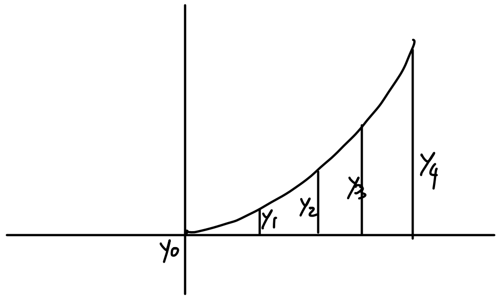
Ex 2 a): Use Simpson's Rule with $n = 4$ subintervals to approximate $\int_0^2 5x^4\,dx$

Sol:

Subintervals:



$\Delta x = \frac{2-0}{4} = \frac{1}{2}$



Approximation using Simpson's Rule is

$$S = \frac{\Delta x}{3}\left[Y_0 + 4Y_1 + Y_2 \right] + \frac{\Delta x}{3}\left[Y_2 + 4Y_3 + Y_4 \right]$$

$$= \frac{\left(\frac{1}{2}\right)}{3}\left[Y_0 + 4Y_1 + 2Y_2 + 4Y_3 + Y_4 \right]$$

$$= \frac{1}{6}\left[\overbrace{5(0)^4}^{Y_0} + 4\cdot\overbrace{5\left(\tfrac{1}{2}\right)^4}^{Y_1} + 2\cdot\overbrace{5(1)^4}^{Y_2} + 4\cdot\overbrace{5\left(\tfrac{3}{2}\right)^4}^{Y_3} + \overbrace{5(2)^4}^{Y_4} \right]$$

$$= \frac{1}{6}\left[4\,\frac{5}{16} + 2(5) + 4\left(\frac{405}{16}\right) + 80 \right]$$

$$= 32\frac{1}{12}$$

2b): Compare with exact value.

Sol: $\int_0^2 5x^4\,dx = 5\,\frac{x^5}{5}\Big|_0^2 = 2^5 = 32.$

Error : $\frac{1}{12}$. Percentage error : $\left(\frac{1}{12}\right)\Big/32 \times 100\% \approx 0.0026 \times 100\%$

$= 0.26\%$

very small w/ only $n=4$ subintervals.

Thm:

Let M be any upper bound for the values of $|f^{(4)}|$ on $[a,b]$.

4th derivative of f

Then the error E_S in the Simpson's Rule approximation of $\int_a^b f(x)\,dx$ for n steps satisfies

$$|E_S| \leq \frac{M(b-a)^5}{180\,n^4}$$

(much smaller than the upper bound using Trapezoidal Rule)

Ex 2 c):

Find an upper bound for the error in estimating $\int_0^2 5x^4\,dx$ using Simpson's Rule with $n=4$ subintervals.

Sol:

$f(x) = 5x^4$

$f'(x) = 20x^3$

$f''(x) = 60x^2$

$f^{(3)}(x) = 120x$

$f^{(4)}(x) = 120$

$M = 120$

$$|E_S| \leq \frac{120(2-0)^5}{180 \cdot 4^2} = \frac{1}{12}$$

Ex 2d): Estimate the minimum number of subintervals needed to approximate $\int_0^2 5x^4 \, dx$

using Simpson's Rule with an error of magnitude less than 10^{-4}.

Sol: By the thm, if

$$\frac{M(b-a)^5}{180 n^4} < 10^{-4}$$

then the error E_s satisfies $|E_s| < 10^{-4}$, as required.

$M = 120$ from before, $b-a = 2$.

We want n to satisfy

$$\frac{\overset{2}{\cancel{120}}(2)^5}{\underset{3}{\cancel{180}}\, n^4} < 10^{-4}$$

Solve this inequality for n:

$$\frac{2^6}{3} 10^4 < n^4$$

$$\frac{64}{3} 10^4 < n^4$$

$$21.5 \approx \left(\frac{64}{3}\right)^{\frac{1}{4}} 10 < n$$

Since n must be even in Simpson's Rule, we estimate the minimum number of subintervals required to be $\boxed{n = 22}$.