Sec 8.4 Integration of Rational Functions by Partial Fractions

Def A rational function is a ratio of polynomials

Example $\dfrac{9x+3}{x^2-9}$ and $\dfrac{6x^3+3x^2-45x-24}{x^2-9}$ are rational functions

Def A rational function $\dfrac{P(x)}{Q(x)}$ is proper if $\deg(P) < \deg(Q)$, otherwise it is improper.

Example $\dfrac{9x+3}{x^2-9}$ is a proper rational function

$\deg(P) = 1$
$\deg(Q) = 2$

$\dfrac{6x^3+3x^2-45x-24}{x^2-9}$ is an improper rational function

$\deg(P) = 3$
$\deg(Q) = 2$

Long division

- Goal : Integrate *any* proper rational function
- Any improper rational function $\dfrac{P(x)}{Q(x)}$ can be written as

$$\underbrace{\left(\text{a polynomial } S(x)\right)}_{\text{can integrate using power rule}} + \underbrace{\frac{R(x)}{Q(x)}}_{\text{a proper rational function}}$$

Example: • $\dfrac{6x^3 + 3x^2 - 45x - 24}{x^2 - 9}$ is an improper rational function.

- Perform long division to write it as

$$\underbrace{S(x)}_{\text{a polynomial}} + \frac{R(x)}{Q(x)}$$

← deg of R(x) smaller than deg of Q(x)

S(x) will go here

$$\underbrace{x^2 - 9}_{Q(x)} \overline{\big)\, \underbrace{6x^3 + 3x^2 - 45x - 24}_{P(Q)}}$$

$\vdots$

R(x) will go here

Long division $\frac{P(x)}{Q(x)} = \frac{6x^3 + 3x^2 - 45x - 24}{x^2 - 9}$

Goal: $\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$

① $\frac{6x^3}{x^2} = 6x$

$$
\begin{array}{r|l}
 & \boxed{6x} \\
\hline
x^2 - 9 & 6x^3 + 3x^2 - 45x - 24 \\
 & \boxed{6x^3 \qquad\qquad - 54x}
\end{array}
$$

② $\boxed{6x}(x^2 - 9) = 6x^3 - 54x$

Long division $\frac{P(x)}{Q(x)} = \frac{6x^3+3x^2-45x-24}{x^2-9}$

Goal: $\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$

① $\frac{6x^3}{x^2} = 6x$

$$
\begin{array}{r|l}
 & \quad 6x \\
\hline
x^2-9 & 6x^3 + 3x^2 - 45x - 24 \\
 & 6x^3 \qquad\quad - 54x \\
\hline
 & \qquad 3x^2 + 9x - 24
\end{array}
$$

② $6x(x^2-9) = 6x^3 - 54x$

③ Subtract $6x^3+3x^2-45x-24$
with $6x^3 \qquad -54x$

deg is not smaller than deg of $Q(x) = x^2-9$, so keep going

Long division $\dfrac{P(x)}{Q(x)} = \dfrac{6x^3 + 3x^2 - 45x - 24}{x^2 - 9}$

Goal: $\dfrac{P(x)}{Q(x)} = S(x) + \dfrac{R(x)}{Q(x)}$

① $\dfrac{6x^3}{x^2} = 6x$

①′ $\dfrac{3x^2}{x^2} = 3$

$$
\begin{array}{r|l}
 & 6x + \boxed{3} \\
\hline
x^2 - 9 & 6x^3 + 3x^2 - 45x - 24 \\
 & 6x^3 \qquad\qquad - 54x \qquad - \\
\hline
 & 3x^2 + 9x - 24 \\
 & \boxed{3x^2 \qquad\qquad - 27}
\end{array}
$$

② $6x(x^2-9) = 6x^3 - 54x \rightarrow$

③ Subtract $6x^3 + 3x^2 - 45x - 24$ with $6x^3 - 54x$ $\rightarrow$ $3x^2 + 9x - 24$

$\leftarrow$ deg is <u>not</u> smaller than deg of $Q(x) = x^2 - 9$, so keep going

②′ $\boxed{3}(x^2-9) = 3x^2 - 27 \rightarrow$

Long division $\dfrac{P(x)}{Q(x)} = \dfrac{6x^3+3x^2-45x-24}{x^2-9}$

Goal: $\dfrac{P(x)}{Q(x)} = S(x) + \dfrac{R(x)}{Q(x)}$

① $\dfrac{6x^3}{x^2} = 6x$

①' $\dfrac{3x^2}{x^2} = 3$

$$
\begin{array}{r|l}
 & 6x + 3 \\ \hline
x^2-9 & 6x^3 + 3x^2 - 45x - 24 \\
 & \underline{6x^3 \qquad\quad - 54x} \quad - \\
 & \qquad\ \ 3x^2 + 9x - 24 \\
 & \qquad\ \ \underline{3x^2 \qquad\quad - 27} \quad - \\
 & \qquad\qquad\quad \boxed{9x + 3}
\end{array}
$$

② $6x(x^2-9) = 6x^3 - 54x$

③ Subtract $6x^3+3x^2-45x-24$ with $6x^3 \quad -54x$ → $3x^2 + 9x - 24$

deg is **not** smaller than deg of $Q(x) = x^2-9$, so keep going

②' $3(x^2-9) = 3x^2 - 27$

③' Subtract $3x^2+9x-24$ with $3x^2 \quad -27$ → $9x + 3$

deg is smaller than the denominator $Q(x)$.

We're done.

Long division $\dfrac{P(x)}{Q(x)} = \dfrac{6x^3+3x^2-45x-24}{x^2-9}$

Goal: $\dfrac{P(x)}{Q(x)} = \underbrace{S(x)}_{\text{polynomial}} + \underbrace{\dfrac{R(x)}{Q(x)}}_{\text{proper rational function}}$

$6x+3$ ← the polynomial $S(x)$

$$
\begin{array}{r|l}
 & 6x + 3 \\
\hline
x^2-9 & 6x^3 + 3x^2 - 45x - 24 \\
 & 6x^3 - 54x \qquad - \\
\hline
 & 3x^2 + 9x - 24 \\
 & 3x^2 - 27 \qquad - \\
\hline
 & 9x + 3
\end{array}
$$

$9x+3$ ← the remainder $R(x)$

$$\frac{6x^3+3x^2-45x-24}{x^2-9} = \underbrace{\boxed{6x+3}}_{\text{polynomial}} + \underbrace{\frac{\boxed{9x+3}}{x^2-9}}_{\text{proper rational function}}$$

Partial fraction theorem CASE I

If $\dfrac{R(x)}{Q(x)}$ is proper and

$Q(x)$ is a product of <u>distinct</u> linear polynomials
$(a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$,

then

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}$$ for some numbers $A_1, A_2, \ldots, A_k$.

Example:

- $\dfrac{5}{x^2-4} = \boxed{\dfrac{5}{(x-2)(x+2)} = \dfrac{A}{x-2} + \dfrac{B}{x+2} \text{ for some numbers } A, B}$

- $\dfrac{x^2+2x-1}{2x^3+3x^2-2x} = \boxed{\dfrac{x^2+2x-1}{x(2x-1)(x+2)} = \dfrac{A}{x} + \dfrac{B}{2x-1} + \dfrac{C}{x+2} \text{ for some numbers } A, B, C}$

- $\dfrac{4x}{(x-1)^2(x+1)}$ does not apply to this CASE I because the factor $(x-1)$ appears twice! See CASE II

What is the partial fraction decomposition for $\frac{9x+3}{x^2-9}$?

(a) $\frac{A}{x-3} + \frac{B}{x+3}$

$\frac{9x+3}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}$ by Partial Fraction Theorem CASE I

(b) $\frac{Ax+B}{x-3} + \frac{Cx+D}{x+3}$

(c) $\frac{A}{x^2} + \frac{B}{9}$

(d) $\frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{x+3} + \frac{D}{(x+3)^2}$

Ex 1: $\displaystyle\int \frac{6x^3+3x^2-45x-24}{x^2-9}\,dx$ CASE I

Step 0 If deg(numer) $\geq$ deg(denom), perform long division:

$$\frac{6x^3+3x^2-45x-24}{x^2-9} = 6x+3 + \frac{9x+3}{x^2-9}$$ (see prev.)

Step 1 Partial fraction decomposition for $\frac{9x+3}{x^2-9}$

CASE I: Write $\displaystyle\frac{9x+3}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}$

- Multiply both sides by the denominator $\quad 9x+3 = A(x+3) + B(x-3)$
- Write both sides in standard form $a_n x^n + \dots + a_1 x + a_0$ $\quad 9x+3 = x(A+B) + (3A-3B)$

$$\left.\begin{aligned} 9x &= (A+B)x \\ 3 &= 3A-3B \end{aligned}\right\} \Rightarrow \left.\begin{aligned} 9 &= A+B \\ 1 &= A-B \end{aligned}\right\}$$

$$\Rightarrow \left.\begin{aligned} 9 &= A+B \\ B &= A-1 \end{aligned}\right\} \quad \begin{aligned} &9 = A+(A-1) \Rightarrow 10 = 2A \Rightarrow \boxed{A=5} \\ &B = A-1 = 5-1 \Rightarrow \boxed{B=4} \end{aligned}$$

Ready to integrate:

$$\int \frac{6x^3+3x^2-45x-24}{x^2-9}\,dx$$

$$= \int \left[6x+3+\frac{9x+3}{x^2-9}\right] dx$$

$$= \int \left[6x+3+\frac{5}{x-3}+\frac{4}{x+3}\right] dx$$

$$= \boxed{\frac{6x^2}{2} + 3x + 5\ln|x-3| + 4\ln|x+3| + C}$$

Partial fraction theorem CASE II

CASE II

If $\frac{R(x)}{Q(x)}$ is proper and

$Q(x)$ has a factor $(ax+b)$ that is repeated r times,

instead of $\frac{A}{ax+b}$, write $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_r}{(ax+b)^r}$

Example:

- $$\frac{x^5}{(x+2)^2\,x^4} = \underbrace{\frac{A}{x+2} + \frac{B}{(x+2)^2}}_{\text{factor } (x+2) \text{ is repeated twice}} + \underbrace{\frac{C}{x} + \frac{D}{x^2} + \frac{E}{x^3} + \frac{F}{x^4}}_{x \text{ is repeated four times}}$$

- $$\frac{4x}{(x-1)^2(x+1)} = \underbrace{\frac{A}{x-1} + \frac{B}{(x-1)^2}}_{\text{factor } (x-1) \text{ is repeated twice}} + \frac{C}{x+1}$$

- $$\frac{x^3 - x + 1}{x^2(x-1)^3} = \underbrace{\frac{A}{x} + \frac{B}{x^2}}_{x \text{ is repeated twice}} + \underbrace{\frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}}_{x-1 \text{ is repeated three times}}$$

CASE II

What is the partial fraction decomposition for $\frac{5x^3-3x^2-8x-3}{x^4-3x^3}$

(a) $\frac{A}{x^3}+\frac{B}{x-3}$

(b) $\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x-3}$

$$\frac{5x^3-3x^2-8x-3}{x^3(x-3)} = \underbrace{\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}}+\frac{D}{x-3}$$

because factor x has 3 repetitions

(c) $\frac{A}{x}+\frac{B}{x-3}$

(d) $\frac{A}{x^4}+\frac{B}{3x^3}$

CASE II

Ex 2: $$\int \frac{5x^3 - 3x^2 - 8x - 3}{x^4 - 3x^3}\,dx$$

- $\deg(\text{numer}) = 3 < 4 = \deg(\text{denom})$.
  Don't need to do long division.
- Partial fraction decomposition CASE II

$$\frac{5x^3 - 3x^2 - 8x - 3}{x^3(x-3)} = \underbrace{\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3}}_{x \text{ has } 3 \text{ repetitions}} + \frac{D}{x-3}$$

$$5x^3 - 3x^2 - 8x - 3 = A x^2 (x-3) + B x (x-3) + C(x-3) + D x^3$$
$$= A x^3 - A\,3x^2 + B x^2 - B\,3x + Cx - 3C + D x^3$$
$$= x^3 [A + D] + x^2 [-A3 + B] + x[-3B + C] - 3C$$

$$\left.\begin{aligned} 5 &= A + D \\ -3 &= -3A + B \\ -8 &= -3B + C \\ -3 &= -3C \end{aligned}\right\}$$

$5 = 2 + D \Rightarrow \boxed{D = 3}$

$-3 = -3A + 3 \Rightarrow \boxed{A = 2}$

$-8 = -3B + 1 \Rightarrow \boxed{B = 3}$

$\boxed{C = 1}$

Ready to integrate:

$$\int \frac{5x^3 - 3x^2 - 8x - 3}{x^3(x-3)}\,dx$$

$$= \int \left[\frac{2}{x} + \frac{3}{x^2} + \frac{1}{x^3} + \frac{3}{x-3}\right] dx$$

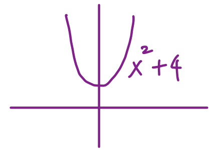
$$= \left(2 \ln|x| - \frac{3}{x} - \frac{1}{2x^2} + 3 \ln|x-3|\right] + C$$

$$= \ln\left(x^2 \left|(x-3)^3\right|\right) - \frac{3}{x} - \frac{1}{2x^2} + C$$

Partial fraction theorem CASE III

A quadratic polynomial ax^2+bx+c has no real root iff $\boxed{b^2-4ac<0,}$

e.g. x^2+4



If $Q(x)$ has a quadratic factor ax^2+bx+c

which has no real root, then

the partial fraction decomposition of $\frac{R(x)}{Q(x)}$ has a term $\frac{Ax+B}{ax^2+bx+c}$

Example:
$$\frac{x}{(x-2)(x^2+1)(x^2+4)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{x^2+4}$$

$$\frac{2x^2-x+4}{x(x^2+4)}=\frac{A}{x}+\frac{Bx+C}{x^2+4}$$

Partial fraction theorem CASE IV

If $Q(x)$ has a quadratic factor ax^2+bx+c (with no real root) which is repeated r times, the partial fraction decomposition of $\frac{R(x)}{Q(x)}$ has the sum

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

Examples:

- $$\frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2} = \frac{A}{x} + \underbrace{\frac{Bx + C}{x^2+1} + \frac{Dx + E}{(x^2+1)^2}}_{\text{two repeats}}$$

- $$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx + D}{x^2+x+1} + \underbrace{\frac{Ex + F}{x^2+1} + \frac{Gx + H}{(x^2+1)^2} + \frac{Ix + J}{(x^2+1)^3}}_{\text{three repeats}}$$

Rationalizing substitutions

Turn a non-rational integrand into a rational integrand by substitution.

Ex 3:

$$\int \frac{1}{x}\frac{1}{\sqrt{x+1}}\,dx = \int \frac{1}{(u^2-1)}\frac{1}{u}\,2u\,du = \int \frac{2}{u^2-1}\,du = \int \frac{2}{(u-1)(u+1)}\,du$$

$\boxed{u=\sqrt{x+1}}$

$u^2 = x+1$

$x = u^2-1$

$\boxed{dx = 2u\,du}$

Alternatively:

$du = \frac{1}{2}\frac{1}{\sqrt{x+1}}\,dx$

$du = \frac{1}{2}\frac{1}{u}\,dx$

$2u\,du = dx$

Partial fraction decomposition case I:

$$\frac{2}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1}$$

Ex 4:

$$\int \frac{34e^x+80}{e^{2x}+7e^x+10}\,dx = \int \frac{34u+80}{u^2+7u+10}\,\frac{1}{u}\,du = \int \frac{34u+80}{u(u+2)(u+5)}\,du$$

$\boxed{u=e^x}$

$\ln u = x$

$\boxed{dx = \frac{1}{u}\,du}$

Alternatively:

$du = e^x\,dx$

$du = u\,dx$

$\frac{1}{u}du = dx$

Partial fraction decomposition case I:

$$\frac{34u+80}{(u)(u+2)(u+5)} = \frac{A}{u} + \frac{B}{u+2} + \frac{C}{u+5}$$