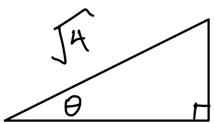
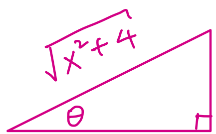
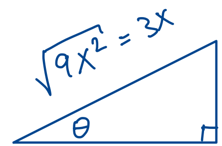
Sec 8.3 Trigonometric Substitutions

Extra (not covered):

$$\int \frac{x^2}{\sqrt{4-x^2}}\,dx \qquad \int x^3\sqrt{x^2+4}\,dx \qquad \int \frac{dx}{\left[9x^2-25\right]^{\frac{3}{2}}} \qquad \int \frac{dx}{\left[x^2+2x+2\right]^2}$$





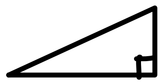


"Complete the square"

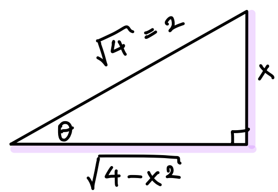
A B C D

Ex A :

- $\displaystyle\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\,dx$ can be solved with Trig substitution

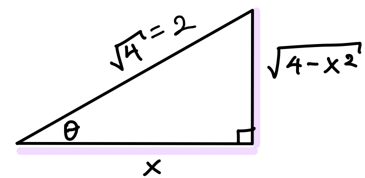
- Label  so that x and $\sqrt{4-x^2}$ are side labels.

Either



$\boxed{\dfrac{x}{2} = \sin\theta}$

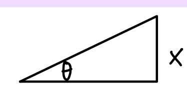
or



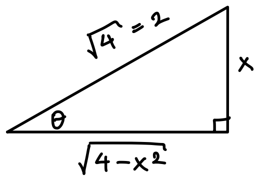
$\boxed{\dfrac{x}{2} = \cos\theta}$

will work

The derivative of $\boxed{\sin\theta}$ is "nicer", so I choose



$\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\,dx$ can be solved with Trig substitution



Let $\frac{x}{2} = \sin\theta$, where

θ is in $\underbrace{\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}$

Domain restriction (Sec 6.6)

so that

$\arcsin\left(\frac{x}{2}\right) = \theta$

makes sense

① $x = \boxed{2\sin\theta}$

$$\boxed{dx = 2\cos\theta \; d\theta}$$

② Write $\frac{1}{\sqrt{4-x^2}}$ in terms of θ:

$$\boxed{\frac{2}{\sqrt{4-x^2}} = \frac{1}{\cos\theta}} \Rightarrow \boxed{\frac{1}{\sqrt{4-x^2}} = \frac{1}{2}\frac{1}{\cos\theta}}$$

③ Write x^2 in terms of θ:

$$\boxed{x^2 = (2\sin\theta)^2}$$

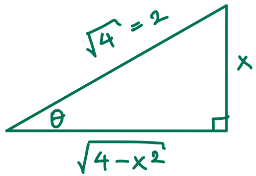
④ Substitute endpoints so that $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$x=0 \Rightarrow \frac{0}{2} = \sin\theta \Rightarrow \theta = 0$

$x=\sqrt{2} \Rightarrow \frac{\sqrt{2}}{2} = \sin\theta \Rightarrow \theta = \frac{\pi}{4}$

A

$$\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\,dx = \int_{\theta=0}^{\theta=\frac{\pi}{4}} 2^2(\sin\theta)^2 \;\frac{1}{2}\,\frac{1}{\cos\theta}\; 2\cos\theta\; d\theta$$



Let $\frac{x}{2} = \sin\theta$, where θ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

① $\boxed{dx = 2\cos\theta\, d\theta}$

② $\boxed{\frac{1}{\sqrt{4-x^2}} = \frac{1}{2}\,\frac{1}{\cos\theta}}$

③ $\boxed{x^2 = (2\sin\theta)^2}$

④ $x=0 \Rightarrow \theta = 0$

$x=\sqrt{2} \Rightarrow \theta = \frac{\pi}{4}$

$$= \int_0^{\frac{\pi}{4}} 4(\sin\theta)^2\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} 4\;\frac{1}{2}\left[1-\cos(2\theta)\right] d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left[2 - 2\cos(2\theta)\right] d\theta$$

$$= 2\theta - 2\,\frac{\sin(2\theta)}{2}\Bigg|_{\theta=0}^{\theta=\frac{\pi}{4}}$$

$$= \left(2\,\frac{\pi}{4} - \sin\left(2\,\frac{\pi}{4}\right)\right) - \left(0 - \sin(0)\right)$$

$$= \frac{\pi}{2} - \sin\left(\frac{\pi}{2}\right)$$

$$= \boxed{\frac{\pi}{2} - 1} \gtrsim \frac{3}{2} - 1 > 0$$

sanity check:

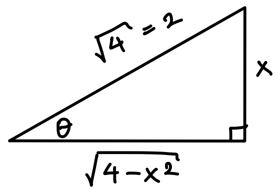
$\frac{x^2}{\sqrt{4-x^2}}$ is positive (except at $x=0$)

so $\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\,dx$

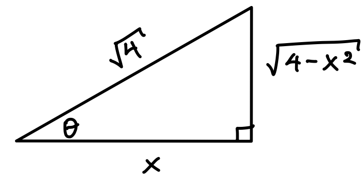
should be positive

Summary so far

Ⓐ $\int \frac{x^2}{\sqrt{4-x^2}}\,dx$ can be solved with Trig substitution



or

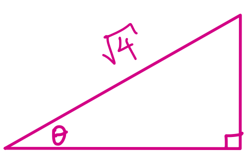


Ⓑ $\int x^3 \sqrt{x^2+4}\,dx$ can also be solved with trig substitution.

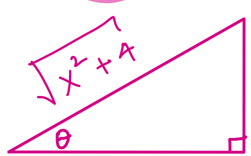
Label  so that x and $\sqrt{4+x^2}$ are side labels.

What should be the label for the hypotenuse?

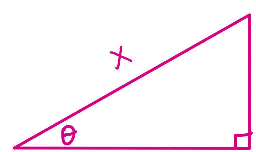
(a)



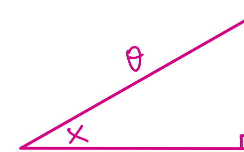
(b)



(c)



(d)

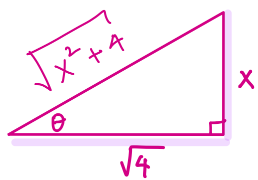


Ex B :

B

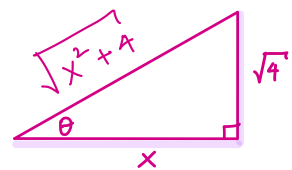
$$\int_0^2 x^3 \sqrt{x^2+4}\; dx$$

Either



$\frac{x}{2} = \tan(\theta)$

or

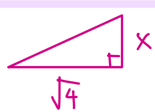


$\frac{x}{2} = \cot(\theta)$

will work

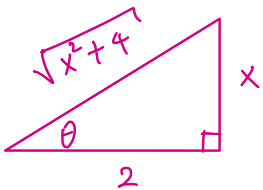
but $\frac{d}{d\theta}\tan(\theta) = \boxed{[\sec(\theta)]^2}$ is easier to work with than $\frac{d}{d\theta}\cot(\theta) = \boxed{-[\csc(\theta)]^2}$

So I choose



B

$$\int_0^2 x^3 \sqrt{x^2+4}\, dx$$



Let $\frac{x}{2} = \tan(\theta)$ where

θ is in $\underbrace{\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$

Domain restriction (Sec 6.6)

so that

$\arctan\left(\frac{x}{2}\right) = \theta$

makes sense

① $x = \boxed{2 \tan(\theta)}$

$$\boxed{dx = 2(\sec(\theta))^2\, d\theta}$$

② Write $\sqrt{x^2+4}$ in terms of θ:

$$\boxed{\frac{\sqrt{x^2+4}}{2} = \frac{1}{\cos(\theta)}}$$

$$\boxed{\sqrt{x^2+4} = \frac{2}{\cos(\theta)}}$$

③ Write x^3 in terms of θ:

$$\boxed{x^3 = 8(\tan\theta)^3}$$

④ Substitute endpoints so that $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$x = 0 \Rightarrow \frac{0}{2} = \tan\theta \Rightarrow \theta = 0$

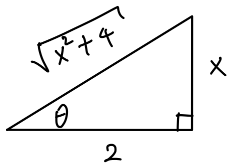
$x = 2 \Rightarrow \frac{2}{2} = \tan\theta \Rightarrow \theta = \frac{\pi}{4}$

B

$$\int_0^2 x^3 \sqrt{x^2+4}\, dx = \int_{\theta=0}^{\theta=\frac{\pi}{4}} 8(\tan\theta)^3 \, \frac{2}{\cos\theta} \, 2(\sec\theta)^2 \, d\theta$$

$$= 32 \int_0^{\frac{\pi}{4}} \underbrace{(\tan\theta)^3 (\sec\theta)^3 \, d\theta}$$

Use technique from Lecture 8.2 notes



Let $\frac{x}{2} = \tan(\theta)$ where

θ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

① $\boxed{dx = 2(\sec(\theta))^2 \, d\theta}$

② $\boxed{\sqrt{x^2+4} = \frac{2}{\cos(\theta)}}$

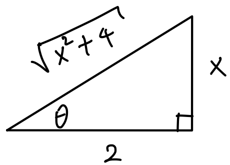
③ $\boxed{x^3 = 8(\tan\theta)^3}$

④ $x = 0 \Rightarrow \theta = 0$

$x = 2 \Rightarrow \theta = \frac{\pi}{4}$

B

$$\int_0^2 x^3\sqrt{x^2+4}\;dx$$



Let $\frac{x}{2} = \tan(\theta)$ where

θ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

① $\boxed{dx = 2(\sec(\theta))^2\,d\theta}$

② $\boxed{\sqrt{x^2+4} = \frac{2}{\cos(\theta)}}$

③ $\boxed{x^3 = 8(\tan\theta)^3}$

④ $x=0 \Rightarrow \theta = 0$

$x=2 \Rightarrow \theta = \frac{\pi}{4}$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{4}} 8(\tan\theta)^3\,\frac{2}{\cos\theta}\,2(\sec\theta)^2\,d\theta$$

$$= 32\int_0^{\frac{\pi}{4}} (\tan\theta)^3(\sec\theta)^3\,d\theta$$

$$= 32\int_0^{\frac{\pi}{4}} (\tan x)^2(\sec x)^2\,\underline{\sec(x)\tan(x)}\,dx$$

put one tan(x) sec(x) aside

because $\frac{d}{dx}\sec(x) = \sec x\tan x$

$$= 32\int_0^{\frac{\pi}{4}} \left[(\sec x)^2 - 1\right](\sec x)^2\,\sec(x)\tan(x)\,dx$$

Apply $(\tan x)^2 = (\sec x)^2 - 1$

$$= 32\int_{u=\sec 0}^{u=\sec\frac{\pi}{4}} \left[u^2-1\right]u^2\,du$$

$u = \sec x$

$du = \sec x\tan x\,dx$

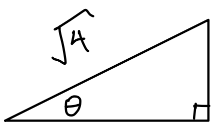
$$= 32\int_{\frac{1}{\cos 0}=1}^{\frac{1}{\cos\frac{\pi}{4}}=\frac{2}{\sqrt{2}}=\sqrt{2}} (u^4 - u^2)\,du$$

$$= 32\left(\frac{u^5}{5} - \frac{u^3}{3}\Bigg|_{u=1}^{u=\sqrt{2}}\right)$$

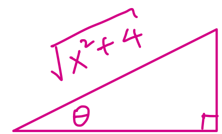
$$= 32\left(\frac{1}{5}2^{\frac{5}{2}} - \frac{1}{5} - \frac{1}{3}2^{\frac{3}{2}} + \frac{1}{3}\right)$$

Summary so far

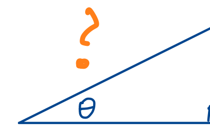
A $$\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\,dx$$



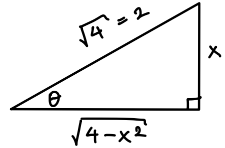
B $$\int_0^{2} x^3\sqrt{x^2+4}\,dx$$



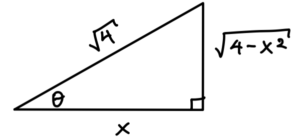
C $$\int \frac{1}{\left[9x^2-25\right]^{\frac{3}{2}}}\,dx$$



(A) $\int \frac{x^2}{\sqrt{4-x^2}}\,dx$

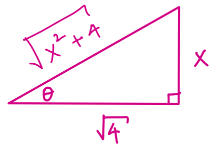


or

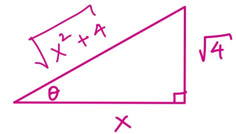


will work

(B) $\int x^3 \sqrt{x^2+4}\,dx$



or



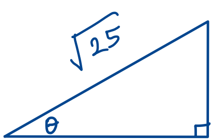
will work

(C) $\int \frac{1}{\left[9x^2-25\right]^{\frac{3}{2}}}\,dx$

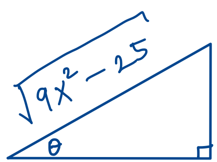
Label  so that x and $\sqrt{9x^2-25}$ are side labels.

What should be the label for the hypotenuse?

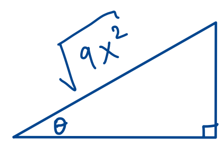
option (a):



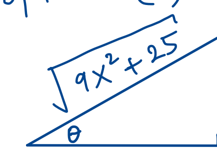
option (b):



option (c):



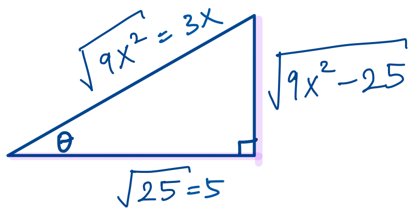
option (d):



Ex (C):

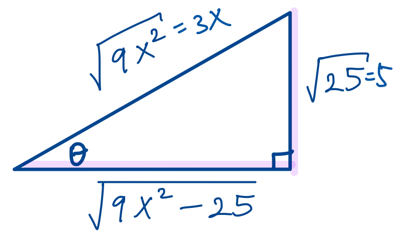
$$\int \frac{1}{\left[9x^2-25\right]^{\frac{3}{2}}}\,dx$$

Either



$$\boxed{\frac{3}{5}x=\sec(\theta)}$$

or

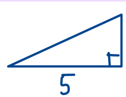


$$\boxed{\frac{3}{5}x=\csc(\theta)}$$

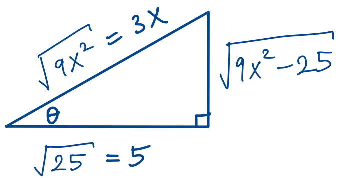
will work

but $\frac{d}{d\theta}\sec(\theta)=\boxed{\sec(\theta)\tan(\theta)}$ is easier to work with than $\frac{d}{d\theta}\csc(\theta)=\boxed{-\csc(\theta)\cot(\theta)}$

so I choose



$$\int \frac{1}{\left[9x^2 - 25\right]^{\frac{3}{2}}}\, dx$$



- Let $\frac{3}{5}x = \sec(\theta)$ where

  θ is in $\underbrace{[0, \frac{\pi}{2}) \text{ or } [\pi, \frac{3\pi}{2})}_{\text{Restricted domain}}$

  (from prev page)

  so that $\operatorname{arcsec}\left(\frac{3}{5}x\right) = \theta$

  makes sense

- $\frac{d}{d\theta} \sec(\theta) = \sec(\theta)\tan(\theta)$

① $x = \boxed{\frac{5}{3}\sec(\theta)}$

$$\boxed{dx = \frac{5}{3}\sec(\theta)\tan(\theta)\, d\theta}$$

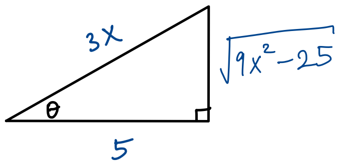
② Write $\frac{1}{\left[9x^2-25\right]^{\frac{3}{2}}}$ in terms of θ:

$$\frac{5}{\sqrt{9x^2-25}} = \frac{1}{\tan\theta}$$

$$\frac{1}{\sqrt{9x^2-25}} = \frac{1}{5}\,\frac{1}{\tan\theta}$$

$$\boxed{\frac{1}{\left[9x^2-25\right]^{\frac{3}{2}}} = \frac{1}{5^3}\,\frac{1}{(\tan\theta)^3}}$$

$$\int \frac{1}{\left[9x^2-25\right]^{\frac{3}{2}}}\,dx = \int \frac{1}{5^3}\frac{1}{(\tan\theta)^3}\,\frac{5}{3}\sec(\theta)\tan(\theta)\,d\theta$$



Let $\frac{3}{5}x = \sec(\theta)$ where

θ is in $[0, \frac{\pi}{2})$ or $[\pi, \frac{3\pi}{2})$

① $\boxed{dx = \frac{5}{3}\sec(\theta)\tan(\theta)\,d\theta}$

② $\boxed{\frac{1}{\left[9x^2-25\right]^{\frac{3}{2}}} = \frac{1}{5^3}\frac{1}{(\tan\theta)^3}}$

$$= \frac{1}{5^2\,3}\int \frac{1}{(\tan\theta)^2}\,\frac{1}{\cos\theta}\,d\theta$$

$$= \frac{1}{75}\int \frac{\cos\theta}{(\sin\theta)^2}\,d\theta$$

$$= \frac{1}{75}\int \frac{1}{u^2}\,du$$

$u = \sin\theta$

$du = \cos\theta\;d\theta$

$$= \frac{1}{75}\left(-\frac{1}{u}\right) + C$$

$$= \frac{1}{75}\left(-\frac{1}{\sin\theta}\right) + C$$

$$= \boxed{\frac{1}{75}\left(-\frac{3x}{\sqrt{9x^2-25}}\right) + C}$$

Summary so far

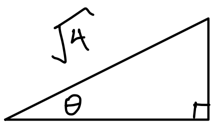
Extra (not covered):

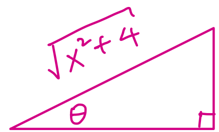
(A) $\displaystyle\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\,dx$

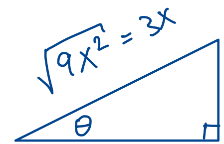
(B) $\displaystyle\int_0^{2} x^3 \sqrt{x^2+4}\;dx$

(C) $\displaystyle\int \frac{dx}{\left[9x^2-25\right]^{\frac{3}{2}}}$

(D) $\displaystyle\int \frac{dx}{\left[x^2+2x+2\right]^2}$







?

Ex D :

$$\int \frac{1}{[x^2+2x+2]^2}\,dx = \boxed{\int \frac{1}{[(x+1)^2+1]^2}\,dx}$$

Step 1 Complete the square

Turn (x^2+2x+2) into $(x+a)^2+b^2$

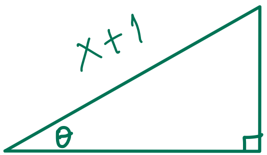
$$\boxed{\begin{aligned}(x^2+2x)+2 &= \left(x^2+2x+\left(\tfrac{2}{2}\right)^2\right)+2-\left(\tfrac{2}{2}\right)^2 \\ &= (x+1)^2+1\end{aligned}}$$

$$\int \frac{1}{[x^2+2x+2]^2}\,dx = \int \frac{1}{[(x+1)^2+1]^2}\,dx$$

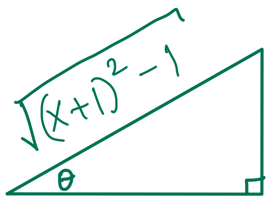
If I want two sides to be labeled $(x+1)$ and $\sqrt{(x+1)^2+1}$

what should be the label for the hypotenuse?

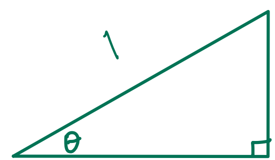
option (a):



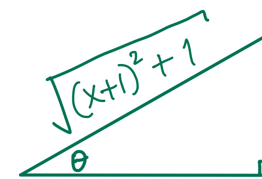
option (b):



option (c):

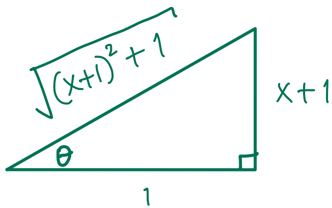


option (d):



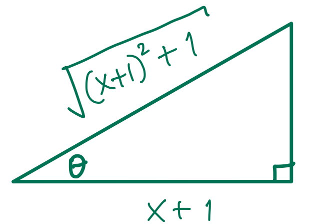
$$\int \frac{1}{[x^2+2x+2]^2}\,dx = \int \frac{1}{[(x+1)^2+1]^2}\,dx$$

Either



$$\boxed{\frac{x+1}{1} = \tan(\theta)}$$

or

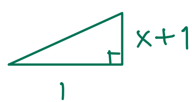


$$\boxed{\frac{x+1}{1} = \cot(\theta)}$$

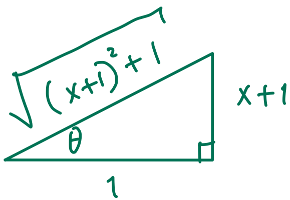
will work

but $\frac{d}{d\theta}\tan(\theta) = [\sec(\theta)]^2$ is easier to work with than $\frac{d}{d\theta}\cot(\theta) = -\csc(\theta)$

So I choose



$$\int \frac{1}{[x^2+2x+2]^2}\,dx = \int \frac{1}{[(x+1)^2+1]^2}\,dx$$



Let $\frac{x+1}{1} = \tan(\theta)$

where θ is in $\underbrace{\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$

Restricting the domain of $\tan(\theta)$ so that $\arctan\left(\frac{x+1}{1}\right) = \theta$ makes sense

① $x = \boxed{-1 + \tan(\theta)}$

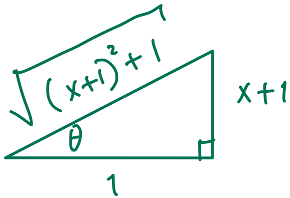
$$\boxed{dx = [\sec(\theta)]^2\, d\theta}$$

② Write $\frac{1}{[(x+1)^2+1]^2}$ in terms of θ:

$$\frac{1}{\sqrt{(x+1)^2+1}} = \cos(\theta)$$

$$\boxed{\frac{1}{[(x+1)^2+1]^2} = [\cos(\theta)]^4}$$

$$\int \frac{1}{[x^2+2x+2]^2}\,dx = \int \frac{1}{[(x+1)^2+1]^2}\,dx = \int [\cos(\theta)]^4\,[\sec(\theta)]^2\,d\theta$$



Let $\frac{x+1}{1} = \tan(\theta)$

where θ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

1 $\boxed{dx = [\sec(\theta)]^2\,d\theta}$

2 $\boxed{\frac{1}{[(x+1)^2+1]^2} = [\cos(\theta)]^4}$

$$= \int [\cos(\theta)]^2\,d\theta$$

$$= \int \frac{1}{2}\left(1+\cos(2\theta)\right)d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin(2\theta)}{2}\right] + C$$

$\sin(t)\cos(t) = \frac{1}{2}\sin(2t)$

$$= \frac{1}{2}\Big[\underset{\arctan\left(\frac{x+1}{1}\right)}{\theta} + \underbrace{\sin(\theta)}_{\frac{opp}{hyp}}\underbrace{\cos(\theta)}_{\frac{adj}{hyp}}\Big] + C$$

$$= \boxed{\frac{1}{2}\left[\arctan(x+1) + \frac{x+1}{(x+1)^2+1}\right] + C}$$

MML Ex

$$\int \frac{1}{\sqrt{t} + 4t\sqrt{t}}\,dt = \int \frac{1}{\sqrt{t}} \frac{1}{1+4t}\,dt$$

$$u = 2\sqrt{t} \Rightarrow 1 + 4t = 1 + u^2$$

$$du = 2\,\frac{1}{2}\,t^{-\frac{1}{2}}\,dt$$

$$du = \frac{1}{\sqrt{t}}\,dt$$

$$= \int \frac{1}{1+u^2}\,du$$

$$= \arctan u$$

$$= \arctan(2\sqrt{t})$$