Sec 8.2 Trig Integrals

Memorize $1 = \sin^2 x + \cos^2 x$ $\qquad \frac{1}{\cos^2 x} = \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} \iff (\sec x)^2 = (\tan x)^2 + 1$

(1.) product of sin & cos w/ "odd cos"

1. Save $(\cos x)$ because $\frac{d}{dx} \sin x = \cos x$
2. Turn all other $\cos x$ into $\sin x$ by $1 = \sin^2 x + \cos^2 x \iff \cos^2 x = 1 - \sin^2 x$
3. Sub $u = \sin x$, $du = \cos x \, dx$

(Same strategy for "odd sin", but swap sin & cos)

(2.) Even cos or sin

Apply $\cos^2 x = \frac{1 + \cos 2x}{2}$ or $\sin^2 x = \frac{1 - \cos 2x}{2}$

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product of sec & tan with...

(4.) "even sec":

1. Save $(\sec x)^2$ because $\frac{d}{dx} \tan x = (\sec x)^2$
2. Turn all other $(\sec x)^2$ into $1 + (\tan x)^2$
3. Sub $u = \tan x$, $du = (\sec x)^2 \, dx$

(3.) "odd tan"

1. Save $\sec x \tan x$ since $\frac{d}{dx} \sec x = \sec x \tan x$
2. Turn all $(\tan x)^2$ into $\sec^2 x - 1$
3. Sub $u = \sec x$, $du = \sec x \tan x \, dx$

Odd powers of cosine

"odd cos"

$$\int (\cos x)^7 \, dx = \int (\cos x)^6 \cos x \, dx$$

- Put one $\cos x$ aside

$$= \int \left[(\cos x)^2\right]^3 \cos x \, dx$$

- convert $(\cos x)^{2k}$ to $\left[(\cos x)^2\right]^k$

$$= \int \left[1 - (\sin x)^2\right]^3 \cos x \, dx$$

- Use $1 = \cos^2 x + \sin^2 x$

$$= \int \left[1 - u^2\right]^3 du$$

- Apply u-sub

  Let $u = \sin x$

  $du = \cos x \, dx$

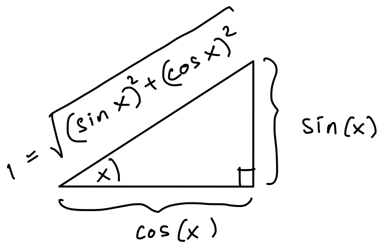
$$= \int (1-u^2)(1-u^2)(1-u^2) \, du$$

- Multiply out

$$= \int \left[1 - 3u^2 + 3u^4 - u^6\right] du$$

$$= u - 3\frac{u^3}{3} + 3\frac{u^5}{5} - \frac{u^7}{7} + C$$

$$= \boxed{(\sin x) - (\sin x)^3 + \frac{3}{5}(\sin x)^5 - \frac{(\sin x)^7}{7} + C}$$



MEMORIZE

$$\boxed{1 = (\cos(x))^2 + (\sin(x))^2}$$

Practice: $\int (\cos x)^3 \, dx$

Textbook Example 1

Odd powers of ~~cosine~~ sine

"odd sin"

$$\int (\sin x)^7\,dx = \int (\sin x)^6 \sin x\,dx$$

~~Put one cos x aside~~

- Put one $\sin x$ aside

$$= \int \left[(\sin x)^2\right]^3 \sin x\,dx$$

- convert $(\sin x)^{2k}$ to $\left[(\sin x)^2\right]^k$

(The same strategy if we replace cosine with sine)

$$= \int \left[1-(\cos x)^2\right]^3 \sin x\,dx$$

- Use $1=\cos^2 x+\sin^2 x$

$$= \int -\left[1-u^2\right]^3\,du$$

- Let $u=\cos x$

  $du=-\sin x\,dx$

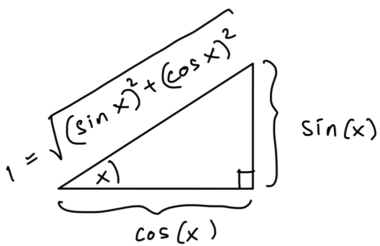
$$= \int -(1-u^2)(1-u^2)(1-u^2)\,du$$

- Multiply out

$$= -\int \left[1-3u^2+3u^4-u^6\right]\,du$$

$$= -u + 3\frac{u^3}{3} - 3\frac{u^5}{5} + \frac{u^7}{7} + C$$

$$= -(\cos x) + (\cos x)^3 - \frac{3}{5}(\cos x)^5 + \frac{(\cos x)^7}{7} + C$$



MEMORIZE

$$\boxed{1 = (\cos(x))^2 + (\sin(x))^2}$$

Product of sine and cosine; Powers are both even

"even sin"

$$(\cos x)^2 = \frac{1 + \cos(2x)}{2}$$

$$(\sin x)^2 = \frac{1 - \cos(2x)}{2}$$

Will be given on exams, but useful to memorize

$$\int (\sin x)^4\,dx = \int \left[(\sin x)^2\right]^2 dx$$

$$= \int \left[\frac{1 - \cos(2x)}{2}\right]^2 dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos(2x) + \left[\cos(2x)\right]^2\right) dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos(2x) + \left[\frac{1 + \cos(4x)}{2}\right]\right) dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos(2x) + \frac{1}{2}\cos(4x)\right) dx$$

$$= \frac{1}{4}\left[\frac{3}{2}x - \frac{2}{2}\sin(2x) + \frac{1}{2}\,\frac{\sin(4x)}{4}\right] + C$$

If both sin & cos are there, may need

$$\sin x \cos x = \frac{1}{2}\sin 2x$$

$$2\sin x \cos x = \sin(2x)$$

Additional Example

"even sin"

$$\int_0^{\frac{\pi}{4}} (\cos x)^2 (\tan x)^2 \, dx$$

$$\int (\cos x)^2 (\tan x)^2 \, dx = \int (\sin x)^2 \, dx$$

$$= \int \frac{1}{2}\left(1 - \cos(2x)\right) dx$$

$$= \frac{1}{2}\left[x - \frac{\sin(2x)}{2}\right] + C$$

$$= \frac{x}{2} - \frac{\sin(2x)}{4} + C$$

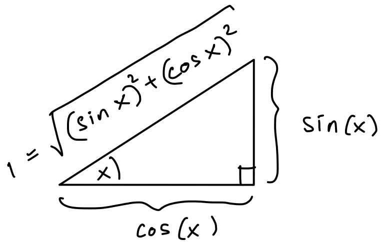
$$\int_0^{\frac{\pi}{4}} (\cos x)^2 (\tan x)^2 \, dx = \left.\frac{x}{2} - \frac{\sin(2x)}{4}\right|_0^{\frac{\pi}{4}}$$

$$= \left[\frac{\pi}{8} - \frac{\sin\left(\frac{\pi}{2}\right)}{4}\right] - \left[0 - \frac{\sin(0)}{4}\right]$$

$$= \boxed{\frac{\pi}{8} - \frac{1}{4}}$$

Product of tangent & secant ; Power of secant is even

"even sec"



MEMORIZE

$$1 = (\cos(x))^2 + (\sin(x))^2$$

$$\frac{1}{(\cos x)^2} = \frac{(\cos(x))^2}{(\cos x)^2} + \frac{(\sin(x))^2}{(\cos x)^2}$$

$$\boxed{(\sec x)^2 = 1 + (\tan x)^2}$$

- $\displaystyle\int (\sec x)^2 \, dx = \boxed{\tan x + C}$ because $\frac{d}{dx}\boxed{\tan(x)} = [\sec(x)]^2$

- $\displaystyle\int (\sec x)^2 (\tan x)^2 \, dx = \int u^2 \, du = \frac{u^3}{3} + C = \boxed{\frac{(\tan x)^3}{3} + C}$

$$\boxed{\begin{aligned} u &= \tan x \\ du &= (\sec x)^2 \, dx \end{aligned}}$$

- $\displaystyle\int (\tan x)^4 \, dx = \int (\tan x)^2 (\tan x)^2 \, dx$

$$= \int [(\sec x)^2 - 1] (\tan x)^2 \, dx$$

Apply $(\tan x)^2 = (\sec x)^2 - 1$

$$= \underbrace{\int (\sec x)^2 (\tan x)^2 \, dx}_{\text{computed above}} - \underbrace{\int (\tan x)^2 \, dx}_{\text{Repeat the same process}}$$

$$= \frac{(\tan x)^3}{3} - \int [(\sec x)^2 - 1] \, dx$$

$$= \boxed{\frac{(\tan x)^3}{3} - (\tan x) + x + C}$$

Ex Like MML

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 8 \quad \csc^4 x \quad dx$$

$\csc(x) = \dfrac{1}{\sin(x)}$

"even csc"

$\dfrac{1}{\sin^2 x} = \dfrac{\sin^2 x}{\sin^2 x} + \dfrac{\cos^2 x}{\sin^2 x} \iff \csc^2 x = 1 + \cot^2 x$

$\int (\csc x)^2 \, dx = -\cot(x) + C$ because $\dfrac{d}{dx} \cot(x) = -(\csc x)^2$

$$\int (\csc x)^2 (\csc x)^2 \, dx = \int [1 + \cot^2 x] (\csc x)^2 \, dx$$

$$= \int \csc^2 x + \cot^2 x \csc^2 x \quad dx$$

Set $u = \cot(x)$

$du = -(\csc x)^2 \, dx$

$-du = (\csc x)^2 \, dx$

Product of tangent & secant ; Power of tangent is odd

"odd tan"

$$\int_0^{\frac{\pi}{4}} (\tan x)^3 (\sec x)^3 \, dx = \int_0^{\frac{\pi}{4}} (\tan x)^2 (\sec x)^2 \underbrace{\sec(x) \tan(x)} \, dx$$

put one $\tan(x)$ $\sec(x)$ aside

because $\frac{d}{dx} \sec(x) = \sec x \tan x$

$$= \int_0^{\frac{\pi}{4}} \left[(\sec x)^2 - 1\right] (\sec x)^2 \sec(x) \tan(x) \, dx$$

Apply
$(\tan x)^2 = (\sec x)^2 - 1$

$$= \int_{u=\sec 0}^{u=\sec \frac{\pi}{4}} \left[u^2 - 1\right] u^2 \, du$$

$u = \sec x$

$du = \sec x \tan x \, dx$

$$= \int_{\frac{1}{\cos 0} = 1}^{\frac{1}{\cos \frac{\pi}{4}} = \frac{2}{\sqrt{2}} = \sqrt{2}} (u^4 - u^2) \, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} \Bigg|_{u=1}^{u=\sqrt{2}}$$

$$= \boxed{\frac{1}{5} 2^{\frac{5}{2}} - \frac{1}{5} - \frac{1}{3} 2^{\frac{3}{2}} + \frac{1}{3}}$$

$$1 = (\cos(x))^2 + (\sin(x))^2$$
$$\frac{1}{(\cos x)^2} \quad \frac{(\cos(x))^2}{(\cos x)^2} \quad \frac{(\sin(x))^2}{(\cos x)^2}$$

$$\boxed{(\sec x)^2 = 1 + (\tan x)^2}$$

Ex like MML

$2\sin\ \cos t = \sin(2t)$

$\sin(6x) = 2\sin(3x)\cos(3x)$

here $t = 3x$

$$\int \cos^5(3x)\,\sin(6x)\,dx$$

$$= \int \cos^5(3x)\, 2\sin(3x)\cos(3x)\,dx$$

$$= 2\int [\cos(3x)]^6 \sin(3x)\,dx$$

$u = \cos(3x)$

$du = -\sin(3x)\,3\,dx$

$-\frac{1}{3}du = \sin(3x)\,dx$

$$= 2\int u^6 \left(-\frac{1}{3}\right) du$$

$$= -\frac{2}{3}\,\frac{u^7}{7} + C$$

$$= \boxed{-\frac{2}{21}[\cos(3x)]^7 + C}$$

The above guidelines are clear-cut. Other cases require different methods.

Sometimes we do u-substitution

- $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$

  Apply u-sub and $\int -\frac{1}{u} \, du = \ln|u|$

  $u = \cos x$

  $du = -\sin x \, dx$

- $\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx$

  Apply u-sub and $\int \frac{1}{u} \, du = \ln|u|$

  $u = \sin x$

  $du = \cos x \, dx$

$\int \sec(x)\,dx$ (I did this Ex during class for Sec 5.5 substitution)

$$\int \sec(x)\,dx = \int \sec(x)\,\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,dx$$

because

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$

$$\frac{d}{dx}\tan(x) = [\sec(x)]^2$$

$$= \int \frac{[\sec(x)]^2 + \sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx$$

$$u = \sec x + \tan x$$

$$du = [\sec x\tan x + (\sec x)^2]\,dx$$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + C$$

$$= \ln|\sec x + \tan x| + C$$

Sometimes we use Integration by Parts

EXAMPLE 6 Evaluate

$$\int \sec^3 x\, dx.$$

Solution We integrate by parts using

$$u = \sec x, \qquad dv = \sec^2 x\, dx, \qquad v = \tan x, \qquad du = \sec x \tan x\, dx.$$

Then

$$\begin{aligned}\int \sec^3 x\, dx &= \sec x \tan x - \int (\tan x)(\sec x \tan x)\, dx \\ &= \sec x \tan x - \int (\sec^2 x - 1) \sec x\, dx \qquad \tan^2 x = \sec^2 x - 1 \\ &= \sec x \tan x + \int \sec x\, dx - \int \sec^3 x\, dx.\end{aligned}$$

Combining the two secant-cubed integrals gives

$$2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx$$

and

$$\int \sec^3 x\, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C.$$