Sec 8.1 Integration by Parts

Differentiation		Integration
Chain rule	$\longleftrightarrow$	u-substitution
product rule	$\longleftrightarrow$	Integration by parts

$$u v' + v u' = (uv)'$$

$$\int u v' \, dx + \int v u' \, dx = \int (uv)' \, dx$$

$$\int u \, dv + \int v \, du = uv$$

Write $v'(x)\,dx = dv$
$u'(x)\,dx = du$

$$\boxed{\int u \, dv = uv - \int v \, du}$$

Example #1 $\int x \cos(x)\, dx = ?$

$$\boxed{\int u\, dv = uv - \int v\, du}$$

Think:

- Can we do u-substitution?

  x and $\cos(x)$ are "unrelated" by differentiation, so u-sub won't work

- Try Integration by Parts

Pick

u	dv
du	v

so that: $\int v\, du$ is simpler than (or at least not more complicated) the original integral

If I pick

$u = \cos(x)$	$dv = x\, dx$
$du = -\sin(x)$	$v = \frac{x^2}{2}$

then $\int v\, du = \underbrace{\int \frac{x^2}{2}\left(-\sin(x)\right)\, dx}_{\text{more complicated than the original } \int x \cos(x)\, dx}$

So I try

$u = x$	$dv = \cos x\, dx$
$du = dx$	$v = \sin x$

then $\int v\, du = \underbrace{\int \sin(x)\, dx}_{\text{we know how to solve :)}}$

(con't to next page)

Example #1 $\int x\cos(x)\,dx = ?$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Pick

$u = x$	$dv = \cos x\,dx$
$du = dx$	$v = \sin x$

$$\int x\cos(x)\,dx = uv - \int v\,du$$

$$= x\sin x - \int \sin x\,dx$$

$$= \boxed{x\sin x + \cos x + C}$$

Similar to Example #1:

$$\int x \sin(x)\, dx = \,?$$

Try at 

$$\boxed{\int u\, dv = uv - \int v\, du}$$

$u = x$	$dv = \sin x\, dx$
$du = dx$	$v = -\cos x$

$$\int x \sin x\, dx = \int \overbrace{x}^{u}\ \overbrace{\sin x\, dx}^{dv} = \overbrace{x}^{u}\ \overbrace{(-\cos x)}^{v} - \int \overbrace{(-\cos x)}^{v}\ \overbrace{dx}^{du}$$

$$= -x \cos x + \int \cos x\, dx$$

$$= -x \cos x + \sin x + C$$

Example #2 Evaluate $\int e^x \sin x \, dx$

Think: u-substitution does not work because e^x and $\sin x$ are "unrelated".

So try Integration by Parts.

Things to try

a) Apply Integration by Parts with

$u = \sin x$	$dv = e^x \, dx$
$du = \cos x \, dx$	$v = e^x$

b) Apply Integration by Parts with

$u = e^x$	$dv = \sin(x) \, dx$
$du = e^x \, dx$	$v = -\cos(x)$

⇦ Will also work. Try option (b) on your own.

c) Integration by Parts won't work ?

Example#2 Evaluate $\int e^x \sin x \, dx$

Try

$u = \sin x$	$dv = e^x\,dx$
$du = \cos x\,dx$	$v = e^x$

- $\int v\,du = \int e^x \cos x \, dx$ is not simpler, but it's not more complicated, so it's OK.

$$\int e^x \sin x \, dx = uv - \int v\,du$$

$$= (\sin x)e^x - \int e^x \cos x \, dx$$

$u = \cos x$	$dv = e^x\,dx$
$du = -\sin x\,dx$	$v = e^x$

$$= (\sin x)e^x - \left[uv - \int v\,du\right]$$

$$= (\sin x)e^x - \left[(\cos x)e^x - \int e^x(-\sin x)\,dx\right]$$

$$= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

$$\int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

$$2\int e^x \sin x \, dx = e^x \sin x - e^x \cos x$$

$$\int e^x \sin x \, dx = \frac{1}{2}e^x\left[\sin x - \cos x\right] + C$$

Example #3

Find $\int \ln(x)\, dx$

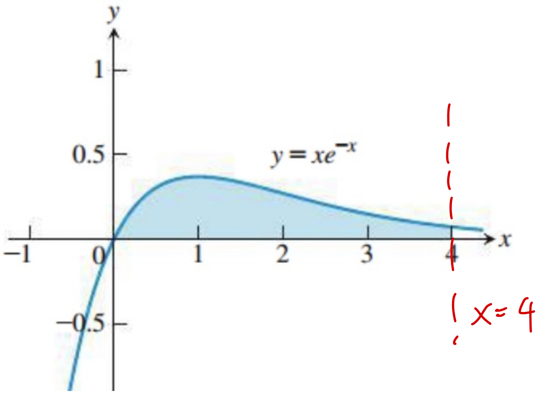
$u = \ln(x)$	$dv = dx$
$du = \frac{1}{x}\, dx$	$v = x$

$$\int \ln(x)\, dx = \underbrace{x}_{v}\, \underbrace{\ln(x)}_{u} - \int \underbrace{x}_{v}\, \underbrace{\frac{1}{x}\, dx}_{du}$$

$$= x \ln(x) - \int 1\, dx$$

$$= \boxed{x \ln(x) - x + C}$$

Example #4 Find the area of the region sketched below



Sol: $\displaystyle\int_0^4 x\, e^{-x}\, dx = \Big[-xe^{-x}\Big]_0^4 - \int_0^4 -e^{-x}\, dx$

$u = x$	$dv = e^{-x}\, dx$
$dv = dx$	$v = -e^{-x}\, dx$

$$= -4e^{-4} + \int_0^4 e^{-x}\, dx$$

$$= -4e^{-4} + \Big[-e^{-x}\Big]_0^4$$

$$= -4e^{-4} + \Big[-e^{-4} - (-e^{0})\Big]$$

$$= -4e^{-4} - e^{-4} + 1$$

$$= \boxed{1 - \frac{5}{e^4}}$$

$e \approx 2.718$, so $e^4 > 2^4 = 16$, so $1 - \frac{5}{e^4}$ is positive.

Ex #5:

$$\int \sin(\sqrt{x})\,dx$$

Hint: First substitute $u = \sqrt{x}$

$u = \boxed{\sqrt{x}}$

$du = \frac{1}{2}\frac{1}{\sqrt{x}}\,dx \Rightarrow \quad du = \frac{1}{2u}\,dx \Rightarrow \quad 2u\,du = \boxed{dx}$

$$\int \sin(\sqrt{x})\,dx = \int 2u \sin(u)\,du$$

$$= 2\left[-u\cos u + \sin u\right] + C$$

$$= \boxed{2\left[-\sqrt{x}\cos\sqrt{x} + \sin\sqrt{x}\right] + C}$$

From previous example

$u = x$	$dv = \sin x\,dx$
$du = dx$	$v = -\cos x$

$$\int x\sin x\,dx = \int \overbrace{x}^{u}\ \overbrace{\sin x\,dx}^{dv} = \overbrace{x}^{u}\ \overbrace{(-\cos x)}^{v} - \int \overbrace{(-\cos x)}^{v}\ \overbrace{dx}^{du}$$

$$= -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + C$$

Ex #6:

$$\int x^{11}\sqrt{x^6+3}\;dx$$

$$u = x^6 \Rightarrow u^{\frac{1}{6}} = x \Rightarrow \boxed{u^{\frac{11}{6}} = x^{11}}$$

$$du = 6x^5\,dx \qquad du = 6\left(u^{\frac{1}{6}}\right)^5 dx \qquad \frac{du}{6u^{\frac{5}{6}}} = dx$$

$$\int \boxed{u^{\frac{11}{6}}}\sqrt{u+3}\,\frac{1}{6u^{\frac{5}{6}}}\,du = \int \frac{1}{6}u^{\frac{11}{6}-\frac{5}{6}}\sqrt{u+3}\;du$$

$$= \frac{1}{6}\int u\sqrt{u+3}\;du$$

$$w = u+3 \Rightarrow w-3 = u$$

$$dw = du$$

$$= \frac{1}{6}\int (w-3)\,w^{\frac{1}{2}}\,dw$$

$$= \frac{1}{6}\int w^{\frac{3}{2}} - 3w^{\frac{1}{2}}\;dw$$

$$= \frac{1}{6}\left[\frac{2}{5}w^{\frac{5}{2}} - 3\frac{2}{3}w^{\frac{3}{2}}\right] + C$$

$$= \frac{2}{30}(u+3)^{\frac{5}{2}} - \frac{2}{6}(u+3)^{\frac{3}{2}} + C$$

$$= \boxed{\frac{1}{15}\left(x^6+3\right)^{\frac{5}{2}} - \frac{1}{3}\left(x^6+3\right)^{\frac{3}{2}} + C}$$