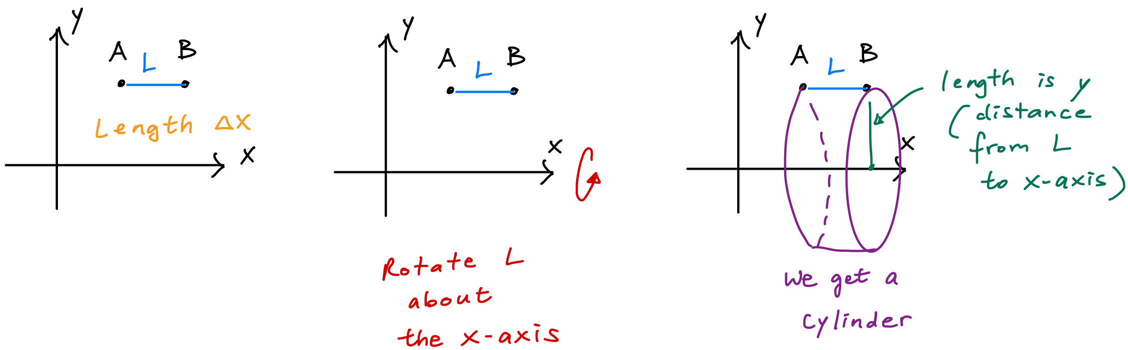
6.4 Areas of surfaces of revolution



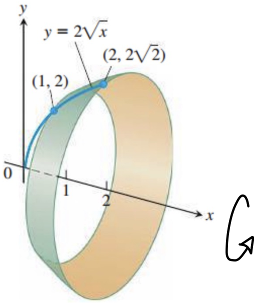
Surface area or lateral (side) surface area of this cylinder is equal to the area of the rectangle

which is $2\pi y \;\Delta x$

Def If $f(x) \geq 0$ and $f'(x)$ is continuous on $[a, b]$, the area of the surface generated by revolving the graph of $y = f(x)$ about the x-axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2}\, dx$$

Ex 1: Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \le x \le 2$, about the x-axis.



Sol: $$S = \int_1^2 2\pi \, y \sqrt{1+\left(\frac{dy}{dx}\right)^2} \, dx$$

$$y = 2x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 2 \, \frac{1}{2} \, x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{x}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{1}{x}} = \sqrt{\frac{x+1}{x}}$$

$$S = \int_1^2 2\pi \left(2\cancel{\sqrt{x}}\right) \frac{\sqrt{x+1}}{\cancel{\sqrt{x}}} \, dx$$

$$= 4\pi \int_1^2 \sqrt{x+1} \, dx$$

$u = x+1$

$du = dx$

$u(1) = 1+1 = 2$

$u(2) = 2+1 = 3$

$$= 4\pi \int_2^3 \sqrt{u} \, du$$

$$= 4\pi \left[\frac{2}{3} u^{\frac{3}{2}}\right]_{u=2}^{u=3}$$

$$= 4\pi \, \frac{2}{3} \left[3^{\frac{3}{2}} - 2^{\frac{3}{2}}\right]$$

$$= \boxed{\frac{8\pi}{3}\left(3\sqrt{3} - 2\sqrt{2}\right)}$$ a positive number

Ex 2:

Find the area of the surface generated by revolving

$x = 2\sqrt{9-y}$, $0 \le y \le 8$ about the y-axis.

Let $f(y) = 2\sqrt{9-y}$

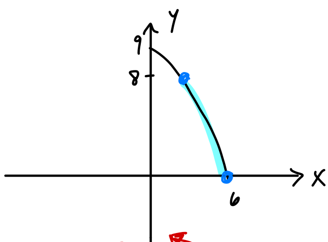
Sol: Sketch first

$x^2 = 4(9-y)$, $x \geqslant 0$

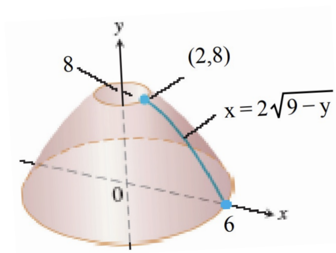
$x^2 = 36 - 4y$

$4y = -x^2 + 36$

$y = -\frac{1}{4}x^2 + 9$, $0 \le y \le 8$



Revolve about y-axis



$$S = \int_{y=0}^{y=8} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^8 2\pi f(y) \sqrt{1 + [f'(y)]^2}\, dy$$

$$x = 2\sqrt{9-y} = 2(9-y)^{\frac{1}{2}}$$

$$\frac{dx}{dy} = 2 \cdot \frac{1}{2}(9-y)^{-\frac{1}{2}}(-1) = -\frac{1}{(9-y)^{\frac{1}{2}}}$$

$$\left(\frac{dx}{dy}\right)^2 = \frac{1}{9-y}$$

(cont Ex 2)

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+\frac{1}{9-y}} = \sqrt{\frac{9-y+1}{9-y}} = \frac{\sqrt{10-y}}{\sqrt{9-y}}$$

$$S = \int_0^8 2\pi \, \overbrace{2\cancel{\sqrt{9-y}}}^{\times} \, \frac{\sqrt{10-y}}{\cancel{\sqrt{9-y}}} \, dy$$

$$= 4\pi \int_0^8 \sqrt{10-y} \, dy$$

$$= 4\pi \left[\frac{2}{3} (10-y)^{\frac{3}{2}} \right]_0^8$$

$$= \frac{8\pi}{3} \left[(10-8)^{\frac{3}{2}} - (10-0)^{\frac{3}{2}} \right]$$

$$= \boxed{\frac{8\pi}{3} \left(2\sqrt{2} - 10\sqrt{10} \right)}$$

is the surface area.