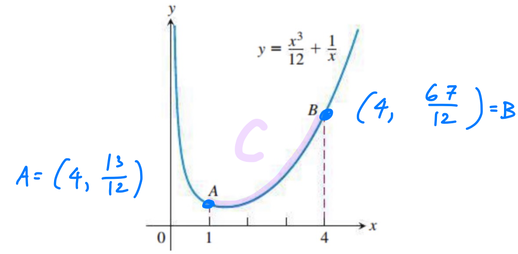
Sec 6.3 Arc length

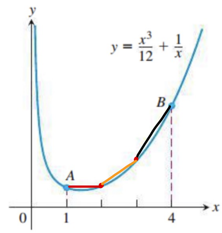
Ex 1: Let C be the curve

$$y = \frac{x^3}{12} + \frac{1}{x}$$

from $x = 1$ to $x = 4$.

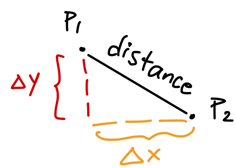


* Take n points on the curve, sum up the distances between adjacent points.



* The limit of this sum as $n \to \infty$ is the <u>length</u> or <u>arc length</u> of C.

* Distance between two points P_1, P_2



is $\sqrt{(\Delta y)^2 + (\Delta x)^2}$ by Pythagorean Thm

$$= \sqrt{(\Delta y)^2 + (\Delta x)^2}\ \frac{1}{\Delta x}\,\Delta x$$ Now multiply by $1 = \frac{\Delta x}{\Delta x}$

$$= \sqrt{\frac{(\Delta y)^2}{(\Delta x)^2} + \frac{(\Delta x)^2}{(\Delta x)^2}}\ \Delta x$$ Bring $\frac{1}{\Delta x}$ inside the radical

$$= \sqrt{\left(\frac{\Delta y}{\Delta x}\right)^2 + 1}\ \Delta x$$

Def If f' is continuous on $[a, b]$,

the length or arc length of the curve $y = f(x)$

from point $A = (a, f(a))$ to $B = (b, f(b))$ is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Sol of previous ex 1:

$$f(x) = \frac{x^3}{12} + \frac{1}{x}, \qquad 1 \le x \le 4$$

$$f'(x) = \frac{1}{12}\, 3x^2 - x^{-2} = \frac{x^2}{4} - x^{-2}$$

$$1 + [f'(x)]^2 = \dots$$

goal: write as $\square^2$ because the arc length formula has $\sqrt{1+(f'(x))^2}$

$$[f'(x)]^2 = \left(\frac{x^2}{4} - x^{-2}\right)^2$$

$$= \frac{x^4}{4^2} - 2\,\frac{x^2}{4}\,x^{-2} + x^{-4}$$

$$= \frac{x^4}{4^2} - \frac{1}{2} + x^{-4}$$

$$1 + [f'(x)]^2 = 1 + \frac{x^4}{4^2} - \frac{1}{2} + x^{-4}$$

$$= \frac{x^4}{4^2} + \frac{1}{2} + x^{-4}$$

$$= \left(\frac{x^2}{4}\right)^2 + 2\,\frac{x^2}{4}\,x^{-2} + \left(x^{-2}\right)^2$$

$$= \left[\frac{x^2}{4} + x^{-2}\right]^2$$

Continue w/ previous ex 1:

$$\sqrt{1+[f'(x)]^2} = \frac{x^2}{4} + x^{-2}$$

Check:
make sure this is ≥ 0
on x in $[1,4]$

$$L = \int_a^b \sqrt{1+[f'(x)]^2}\, dx = \int_1^4 \left(\frac{x^2}{4} + x^{-2}\right) dx$$

$$= \left[\frac{1}{4}\frac{x^3}{3} + \frac{x^{-1}}{-1}\right]_1^4 = \left(\frac{1}{12}4^3 - \frac{1}{4}\right) - \left(\frac{1}{12}1^3 - \frac{1}{1}\right)$$

$$= \frac{64}{12} - \frac{1}{4} - \frac{1}{12} + 1$$

$$= \frac{72}{12} = \boxed{6}$$

Length of this curve is 6.

Ex 2: Find the length of the curve

$$x = \frac{1}{2}\left(e^{y} + e^{-y}\right), \quad 0 \le y \le 2.$$

Sol:

Length is $$L = \int_{y=0}^{y=2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$\frac{dx}{dy} = \frac{1}{2}\left(e^{y} - e^{-y}\right)$$

$$\left(\frac{dx}{dy}\right)^2 = \left(\frac{1}{2}\left(e^{y} - e^{-y}\right)\right)^2$$

$$= \frac{1}{4}\left(e^{2y} - 2e^{y}e^{-y} + e^{-2y}\right)$$

$$= \frac{1}{4}\left(e^{2y} - 2 + e^{-2y}\right)$$

$$1 + \left(\frac{dx}{dy}\right)^2 = \frac{4}{4} + \frac{1}{4}\left(e^{2y} - 2 + e^{-2y}\right)$$

Goal: write as $\square^2$

$$= \frac{1}{4}\left(e^{2y} + 2 + e^{-2y}\right)$$

$$= \frac{1}{4}\left(e^{y} + e^{-y}\right)^2$$

$$= \left[\frac{1}{2}\left(e^{y} + e^{-y}\right)\right]^2$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \frac{1}{2}\left(e^{y} + e^{-y}\right)$$

Make sure this value is ≥ 0 for y in $[0,2]$

$$L = \int_0^2 \frac{1}{2} e^{y} + e^{-y} = \frac{1}{2}\left[e^{y} + \frac{e^{-y}}{-1}\right]_0^2 = \frac{1}{2}\left[e^{2} - e^{-2} - \left(e^{0} - e^{0}\right)\right]$$

$$= \boxed{\frac{1}{2}\left(e^{2} - \frac{1}{e^{2}}\right)}$$

positive number