Sec 6.2 Volumes using cylindrical shells

EXAMPLE 1 The region enclosed by the x-axis and the parabola $y = f(x) = 3x - x^2$ is revolved about the vertical line $x = -1$ to generate a solid (see Figure 6.16). Find the volume of the solid.

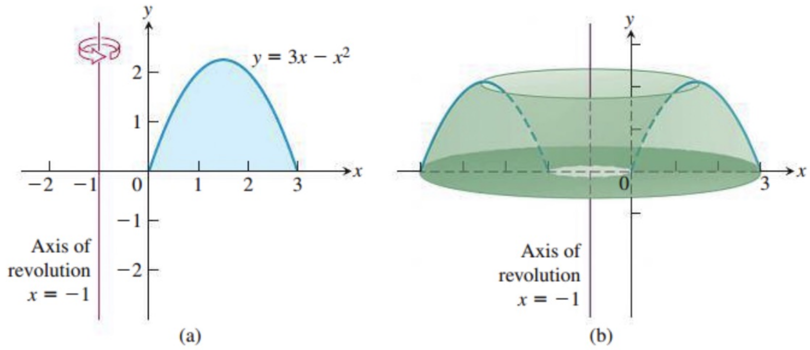
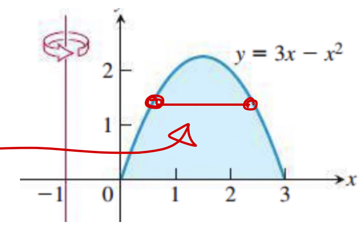


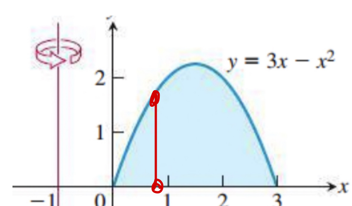
FIGURE 6.16 (a) The graph of the region in Example 1, before revolution. (b) The solid formed when the region in part (a) is revolved about the axis of revolution $x = -1$.

* Using the washer method (Sec 6.1) would be awkward because we would need to write the inner and outer radii of the washer in terms of y,

Draw line segment perpendicular to $x = -1$.

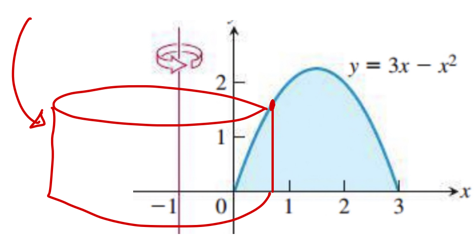


* Instead, draw a line segment parallel to the axis of revolution:



* Then rotate this strip around the axis of revolution:

We get a cylindrical shell, for each x from 0 to 3.



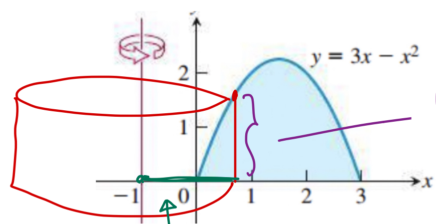
Shell formula for revolution about a vertical line:

Let S be the solid generated by revolving the region between the x-axis and $y = f(x) \geq 0$, $L \leq a \leq b$ about a vertical line $x = L$ (possibly the y-axis).

The volume of S is

$$V = \int_a^b 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dx$$

For Ex 1, we get $V = \displaystyle\int_0^3 2\pi\,(x+1)(3x - x^2)\,dx$



height is $y = 3x - x^2$

radius is $x + 1$

$$= 2\pi \int_0^3 \left(3x^2 - x^3 + 3x - x^2\right) dx$$

$$= 2\pi \int_0^3 \left(2x^2 - x^3 + 3x\right) dx$$

$$= 2\pi \left[2\frac{x^3}{3} - \frac{x^4}{4} + \frac{3x^2}{2} \right]_0^3$$

$$= 2\pi \left(2\frac{3^3}{3} - \frac{3^4}{4} + \frac{3(3)^2}{2} \right)$$

$$= 2\pi \left(2(9) - \frac{81}{4} + \frac{27}{2} \right)$$

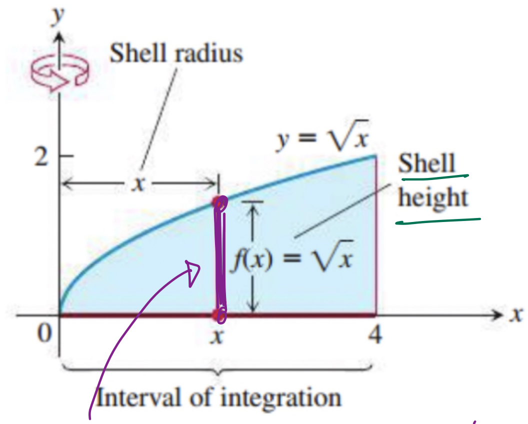
$$= \pi \left(36 - \frac{81}{2} + 27 \right)$$

$$= \boxed{\pi \frac{45}{2}}$$

EXAMPLE 2 The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

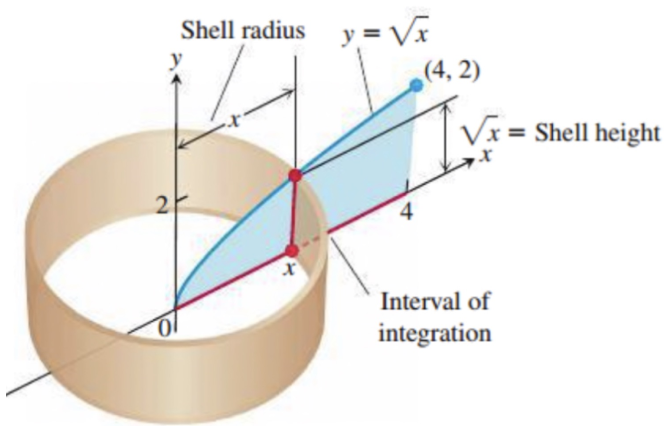
Sol: Sketch of region (required):

axis of revolution is $x=0$ which is vertical



Draw a line segment parallel to the axis of revolution

Sketch of shell (optional):



Volume is $V = \int_0^4 2\pi \, (x) \, (\sqrt{x}) \, dx$

shell radius   shell height

$= 2\pi \int_0^4 x^{\frac{3}{2}} \, dx$ $2^2 \; 2^5 \quad 2^7$

$= 2\pi \left[\frac{2}{5} x^{\frac{5}{2}}\right]_0^4$ $2^4 = 16.16$

$= 2\pi \, \frac{2}{5} \left[(\sqrt{4})^5\right] = \frac{4}{5}\pi \, 2^5 = \boxed{\frac{128\pi}{5}}$

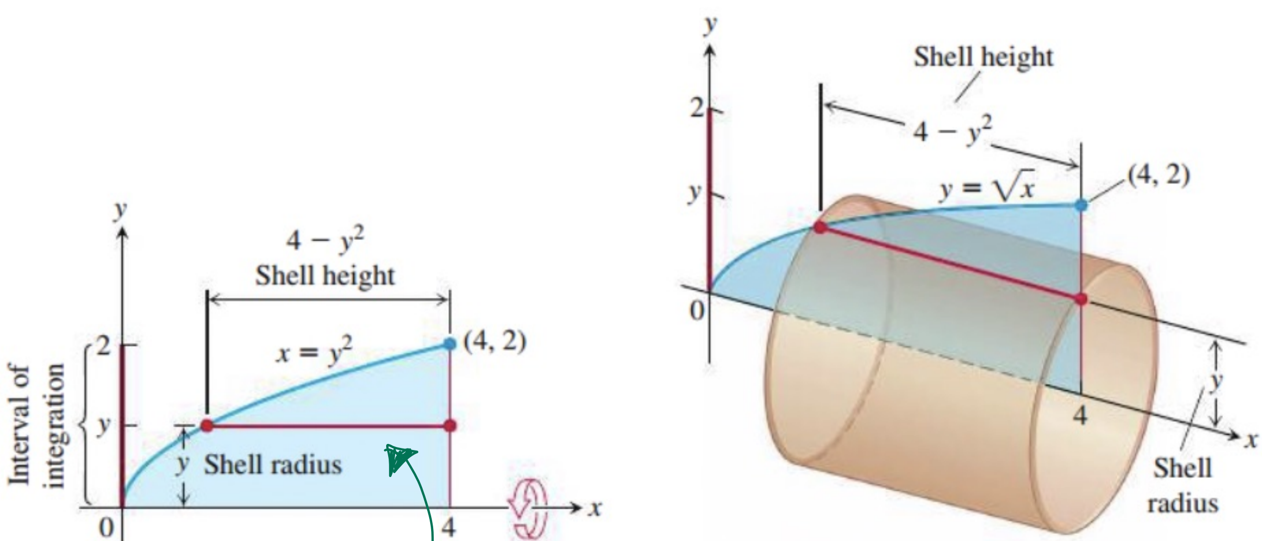
For horizontal axes of revolution, replace "x" with "y".

EXAMPLE 3 The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid by the shell method.

Solution This is the solid whose volume was found by the disk method in Example 4 of Section 6.1. Now we find its volume by the shell method. First, sketch the region and draw a line segment across it *parallel* to the axis of revolution (Figure 6.21a). Label the segment's length (shell height) and distance from the axis of revolution (shell radius). (We drew the shell in Figure 6.21b, but you need not do that.)

In this case, the shell thickness variable is y, so the limits of integration for the shell formula method are $a = 0$ and $b = 2$ (along the y-axis in Figure 6.21). The volume of the solid is

$$\begin{aligned} V &= \int_a^b 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dy \\ &= \int_0^2 2\pi(y)(4 - y^2)\, dy \\ &= 2\pi \int_0^2 (4y - y^3)\, dy \\ &= 2\pi\left[2y^2 - \frac{y^4}{4}\right]_0^2 = 8\pi. \end{aligned}$$



(a) (b)

Region

The shell swept out by the horizontal segment

Summary of the Shell Method

Regardless of the position of the axis of revolution (horizontal or vertical), the steps for implementing the shell method are these.

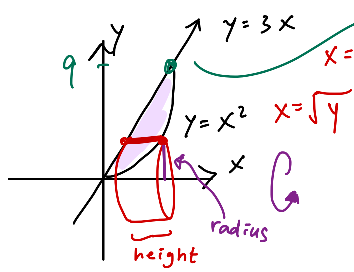
1. *Draw the region and sketch a line segment* across it *parallel* to the axis of revolution. *Label* the segment's height or length (shell height) and distance from the axis of revolution (shell radius).
2. *Find the limits of integration* for the thickness variable.
3. *Integrate* the product 2π (shell radius) (shell height) with respect to the thickness variable (x or y) to find the volume.

* Shell method & washer method give the same answer.
* Both formulas are special cases of a general volume formula using double & triple integrals in Mutivariable Calculus.

More examples using the shell method.

4) Set up the integral to find the volume of the solid generated by rotating the region bounded by $y = x^2$ and $y = 3x$ about the x-axis. Don't solve.

Sol:



intersection point:

$x^2 = 3x$

$x^2 - 3x = 0$

$x(x-3) = 0$

horizontal axis of revolution

$$\int_{y=0}^{y=9} 2\pi \, (y) \left(\sqrt{y} - \frac{y}{3}\right) dy$$

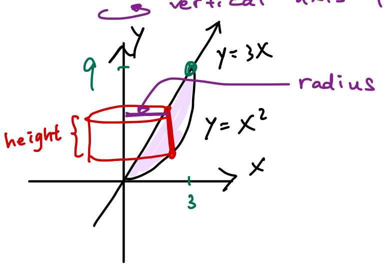
shell radius is vertical distance from axis of revolution

shell height

5) Same region, but rotate about the y-axis.

vertical axis of revolution

Sol:



$$\int_{x=0}^{x=3} 2\pi \, (x) \left(3x - x^2\right) dx$$

shell radius is horizontal distance from axis of revolution

shell height