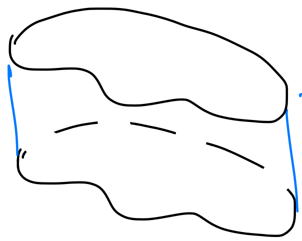
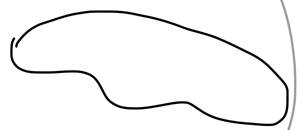
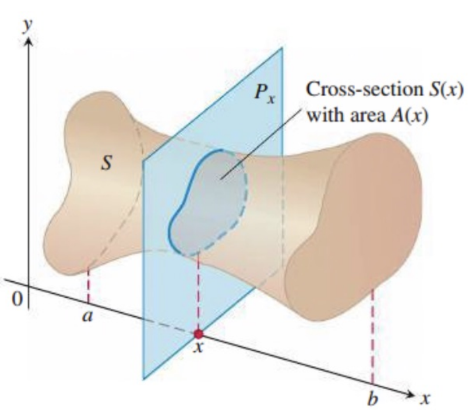
Sec 6.1–6.4: Applications of definite integrals

Sec 6.1 Volumes using cross-sections

(Volume of  height) = (Area of ) · height

$$\text{Volume} = (\text{base area}) \cdot (\text{height})$$

Method 1: Method of slicing



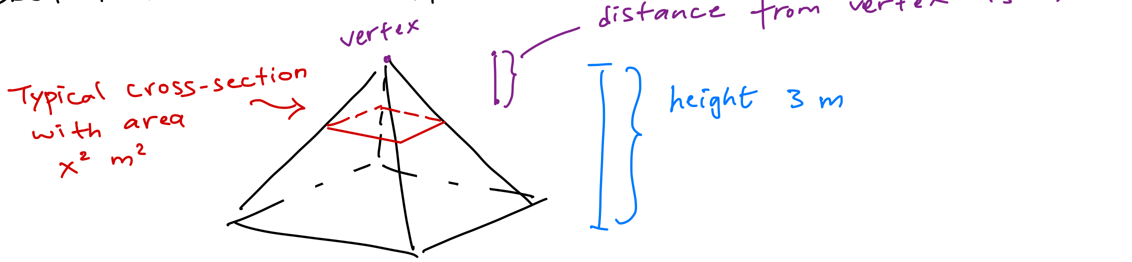
Let S be a solid w/ integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$.

Volume of S is $\int_a^b A(x)\, dx$.

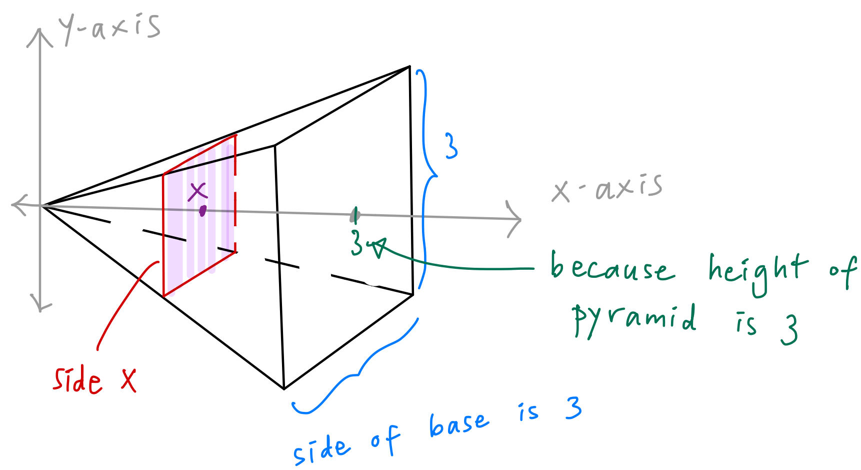
Ex (using method of slicing):

A pyramid 3 m high has a square base that is 3 m on each side. The cross-section of the pyramid perpendicular to the altitude x m down from the vertex is x m on each side. Find the volume.

Step 1 : Sketch the solid & typical cross-section.



Sketch on x-axis



Step 2: Area of cross-section at x is $A(x) = x^2$ because cross-section is a square w/ side x.

Step 3: Limits of integration:

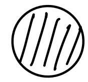
The (square) cross-sections lie on planes from $x=0$ to $x=3$.

Step 4: Integrate to find the volume:

$$V = \int_0^3 A(x)\,dx = \int_0^3 x^2\,dx = \frac{x^3}{3}\Big|_0^3 = \frac{3^3}{3} - \frac{0^3}{3} = 9$$

volume is $9\,m^3$.

Method 2: The disk method for computing a solid of revolution.



filled circle

Ex (for the disk method):

The region between the curve $y = \sqrt{x}$, $0 \le x \le 4$, and the line $y = 0$ is revolved about the x-axis to generate a solid. Find its volume.

Sol:

$$\int_0^4 \begin{pmatrix} \text{area of} \\ \text{cross section} \\ \text{at } x \end{pmatrix} dx$$

$$= \int_0^4 \begin{pmatrix} \text{area of} \\ \text{disk with} \\ \text{radius } \sqrt{x} \end{pmatrix} dx$$

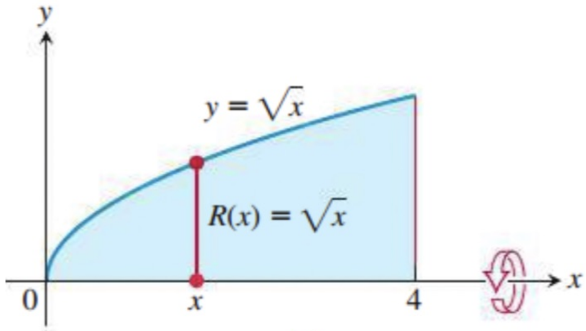
$$= \int_0^4 \pi \,(\text{radius})^2 \, dx$$

$$= \int_0^4 \pi \left(\sqrt{x}\right)^2 dx$$

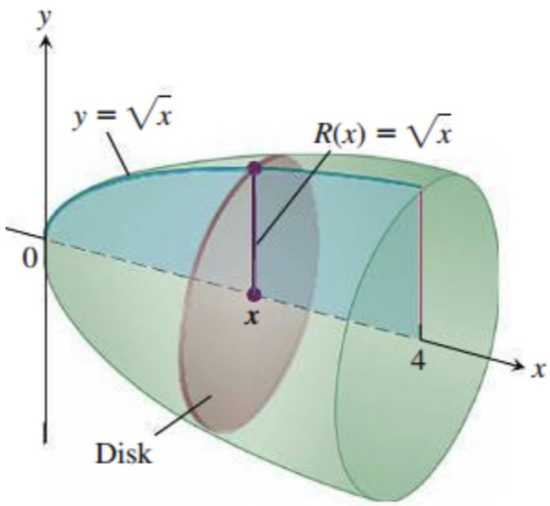
$$= \int_0^4 \pi \, x \, dx$$

$$= \pi \left. \frac{x^2}{2} \right|_0^4$$

$$= \pi \, \frac{16}{2} - 0 = \boxed{\pi \, 8}$$



(a) Region

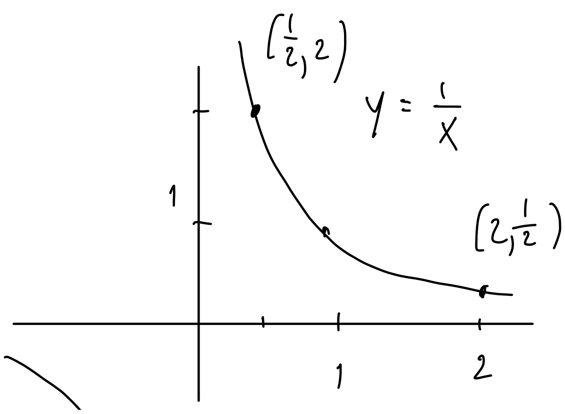


(b) Solid of revolution

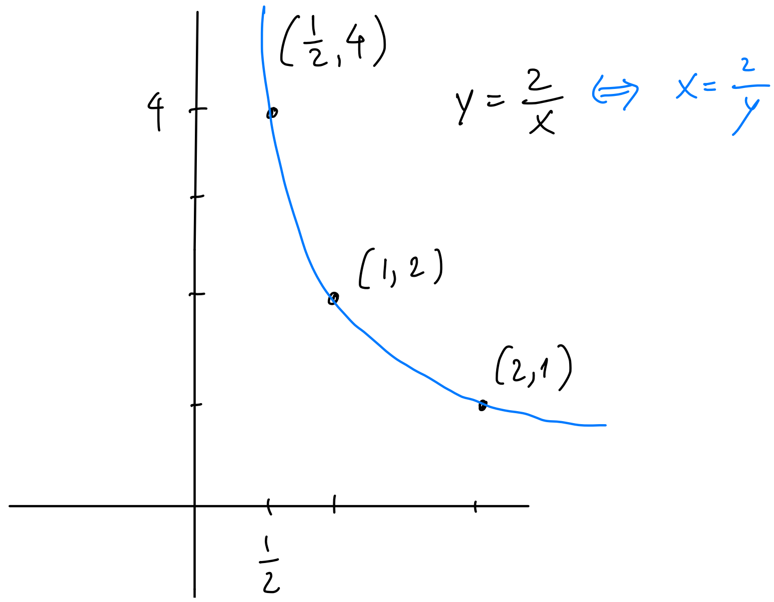
Second Ex (for the disk method):

Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = \frac{2}{y}$, $1 \le y \le 4$, about the y-axis. **Sol:**

$$x = \frac{2}{y} \iff y = \frac{2}{x}$$



$\rightarrow$



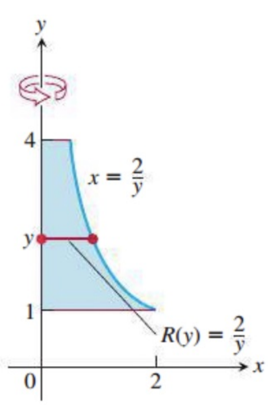
Radius of disk is distance from $x = \frac{2}{y}$ to $x = 0$: $\frac{2}{y}$

$$\text{Volume} = \int_1^4 \pi \underbrace{\left(\frac{2}{y}\right)^2}_{\text{radius of disk}} dy = \int_1^4 \pi \, 4 y^{-2} \, dy$$

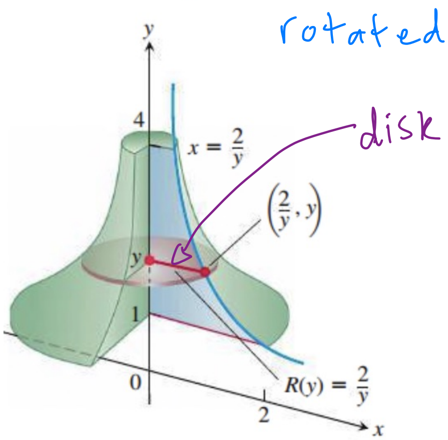
$$= \pi \, 4 \left. \frac{y^{-1}}{-1} \right|_1^4$$

$$= -4\pi \left[\frac{1}{y} \right]_1^4$$

$$= -4\pi \left[\frac{1}{4} - 1 \right] = \boxed{3\pi}$$



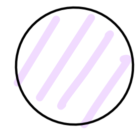
(a) Region which is to be rotated about the y-axis

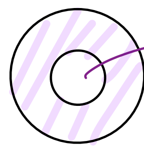


(b)

Method 3: The washer method for computing a solid of revolution.

This method is to be used when

the cross-section is not a disk 

but a washer  this means the solid has a hole

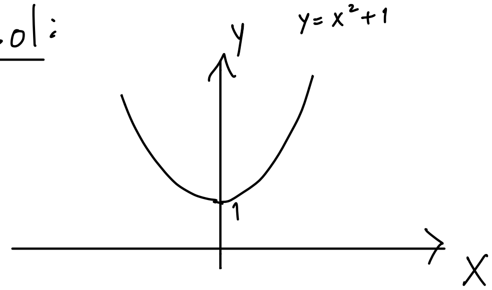
Volume of washer rotated about the x-axis

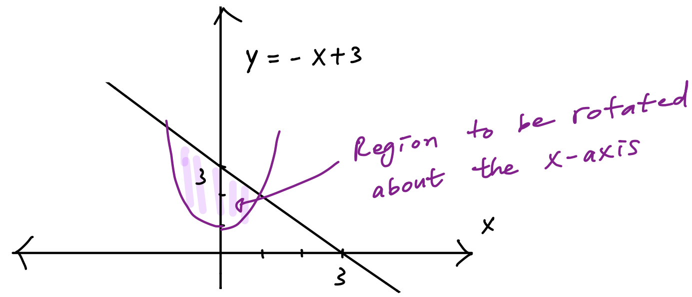
$$\int_a^b \pi\left(\underbrace{(R(x))}_{\substack{\text{outer}\\ \text{radius}\\ \text{of washer}}}^2 - \underbrace{(r(x))}_{\substack{\text{inner}\\ \text{radius}\\ \text{of washer}}}^2\right)\, dx$$

Ex (for the washer method):

The region bounded by the curve $y = x^2+1$ & the line $y = -x+3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Sol:

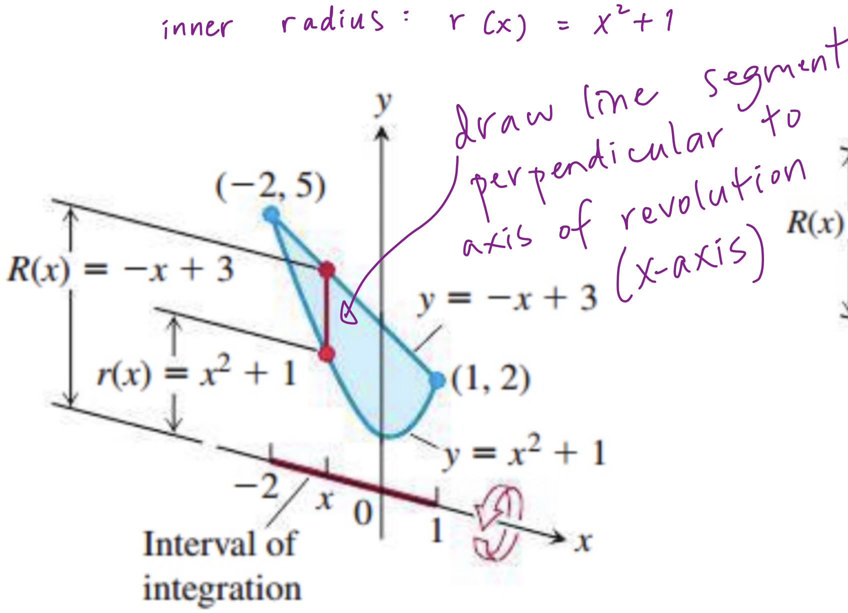


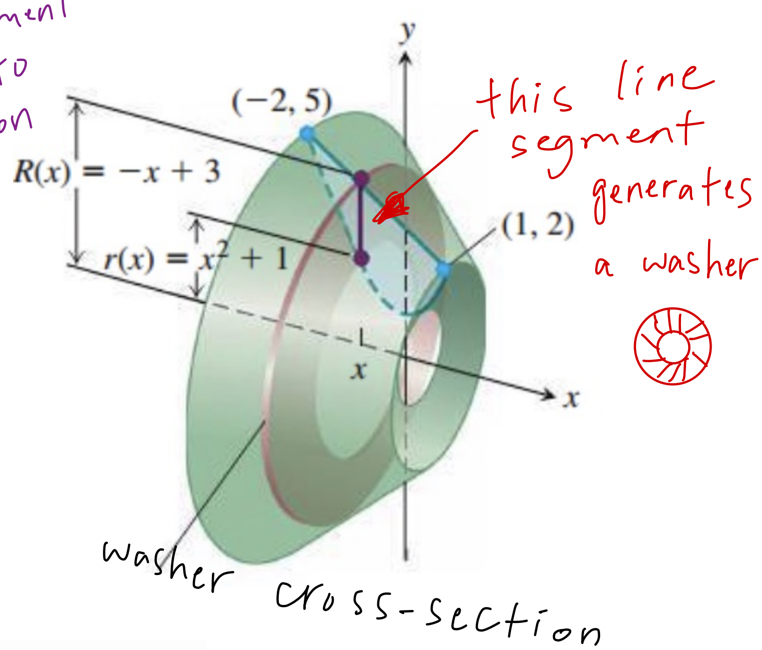


Intersection points: Set $x^2+1 = -x+3 \Rightarrow x^2+x-2=0 \Rightarrow (x+2)(x-1)=0 \Rightarrow x=-2,\ x=1$

outer radius: $R(x) = -x+3$ (because the line has higher y-value than the parabola)

inner radius: $r(x) = x^2+1$





Volume is

$$V = \int_a^b \pi\left([R(x)]^2 - [r(x)]^2\right) dx$$

$$= \int_{-2}^{1} \pi\left((-x+3)^2 - (x^2+1)^2\right) dx$$

$$= \pi \int_{-2}^{1} (8 - 6x - x^2 - x^4)\, dx$$

$$= \pi\left[8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5}\right]_{-2}^{1} = \frac{117\pi}{5}.$$

2nd Ex (for washer method):

EXAMPLE 10 The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid.

axis of revolution

Solution First we sketch the region and draw a line segment across it perpendicular to the axis of revolution (the y-axis). See Figure 6.15a.

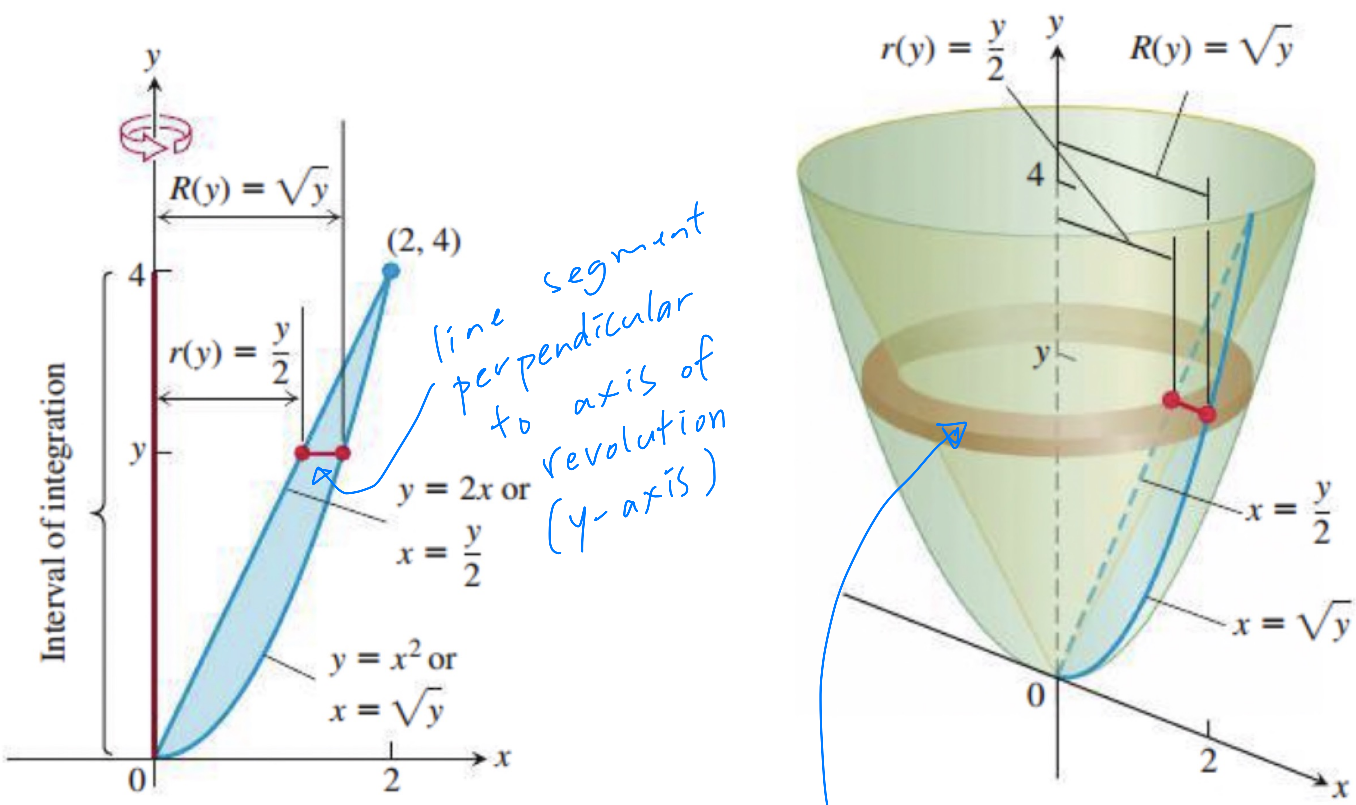
The radii of the washer swept out by the line segment are $R(y) = \sqrt{y}$, $r(y) = y/2$ (Figure 6.15).

The line and parabola intersect at $y = 0$ and $y = 4$, so the limits of integration are $c = 0$ and $d = 4$. We integrate to find the volume:

$$V = \int_c^d \pi\left([R(y)]^2 - [r(y)]^2\right) dy \qquad \text{Rotation around } y\text{-axis}$$

$$= \int_0^4 \pi\left([\sqrt{y}]^2 - \left[\frac{y}{2}\right]^2\right) dy \qquad \text{Substitute for radii and limits of integration.}$$

$$= \pi\int_0^4 \left(y - \frac{y^2}{4}\right) dy = \pi\left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 = \frac{8}{3}\pi.$$



(a)

(b)

line segment perpendicular to axis of revolution (y-axis)

Region to be rotated about the y-axis

The washer swept out by the line segment