Sec 5.6 Definite Integral Substitution & the area between curves

Ex: Evaluate $\displaystyle\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$

Method A (old):

Step 1: Find an antiderivative of $f(x) = 3x^2\sqrt{x^3+1}$

$$\int 3x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\,du$$

We use Substitution Rule for indefinite integral (Sec 5.5)

$$\boxed{\begin{aligned} \text{Try } u &= x^3+1 \\ du &= 3x^2\,dx \end{aligned}}$$

$$= \int u^{\frac{1}{2}}\,du$$

$$= \frac{2}{3}u^{\frac{3}{2}} + C$$

$$= \frac{2}{3}(x^3+1)^{\frac{3}{2}} + C$$

We only need one of the antiderivatives of $f(x) = 3x^2\sqrt{x^3+1}$

so we can take $F(x) = \frac{2}{3}(x^3+1)^{\frac{3}{2}}$ (we choose $C:=0$)

Step 2: $\displaystyle\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = F(x)\Big|_{-1}^{1} = \frac{2}{3}(x^3+1)^{\frac{3}{2}}\Big|_{-1}^{1} = \frac{2}{3}\left[(1^3+1)^{\frac{3}{2}} - (-1+1)^{\frac{3}{2}}\right]$

By FTC Part 2 Sec 5.4

$$= \frac{2}{3}(2)^{\frac{3}{2}}$$

$$= \frac{2}{3}\sqrt{2^3}$$

$$= \boxed{\frac{4}{3}\sqrt{2}}$$

Method B (new)

Apply substitution directly to definite integrals:

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \int_{u=(-1)^3+1}^{u=1^3+1} \sqrt{u}\,du$$

change limits of integration

$$\boxed{\begin{aligned} u &= x^3+1 \\ du &= 3x^2\,dx \end{aligned}}$$

$$= \int_0^2 u^{\frac{1}{2}}\,du$$

$$= \frac{2}{3}u^{\frac{3}{2}}\Big|_{u=0}^{u=2}$$

$$= \frac{2}{3}(2)^{\frac{3}{2}} - \frac{2}{3}(0)^{\frac{3}{2}}$$

$$= \boxed{\frac{4\sqrt{2}}{3}}$$

EXAMPLE 2 We use the method of transforming the limits of integration.

← Like MML# 1

(a)

$$\int_{\pi/4}^{\pi/2} \cot\theta\csc^2\theta\,d\theta = \int_1^0 u\cdot(-du)$$

Let $u = \cot\theta$, $du = -\csc^2\theta\,d\theta$, $-du = \csc^2\theta\,d\theta$.
When $\theta = \pi/4$, $u = \cot(\pi/4) = 1$.
When $\theta = \pi/2$, $u = \cot(\pi/2) = 0$.

$$= -\int_1^0 u\,du$$

$$= -\left[\frac{u^2}{2}\right]_1^0$$

$$= -\left[\frac{(0)^2}{2} - \frac{(1)^2}{2}\right] = \frac{1}{2}$$

(b)

$$\int_{-\pi/4}^{\pi/4} \tan x\,dx = \int_{-\pi/4}^{\pi/4} \frac{\sin x}{\cos x}\,dx$$

$$= -\int_{\sqrt{2}/2}^{\sqrt{2}/2} \frac{du}{u}$$

Let $u = \cos x$, $du = -\sin x\,dx$.
When $x = -\pi/4$, $u = \sqrt{2}/2$.
When $x = \pi/4$, $u = \sqrt{2}/2$.

$$= 0$$

Zero width interval

Ex: Evaluate $\displaystyle\int_{-\ln\sqrt{3}}^{0} \frac{e^x}{1+e^{2x}}\,dx = \int_{u=e^{-\ln\sqrt{3}}}^{u=e^0} \frac{1}{1+u^2}\,du$

Try $u = 1+e^{2x}$

$du = 2$

Try
$$\boxed{\begin{array}{l} u = e^x \\ du = e^x\,dx \\ \dfrac{1}{1+e^{2x}} = \dfrac{1}{1+u^2} \end{array}}$$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{1}{1+u^2}\,du$$

How I got $\frac{1}{\sqrt{3}}$

$$e^{-\ln(\sqrt{3})} = e^{\ln\left((\sqrt{3})^{-1}\right)}$$

$$= 3^{-\frac{1}{2}}$$

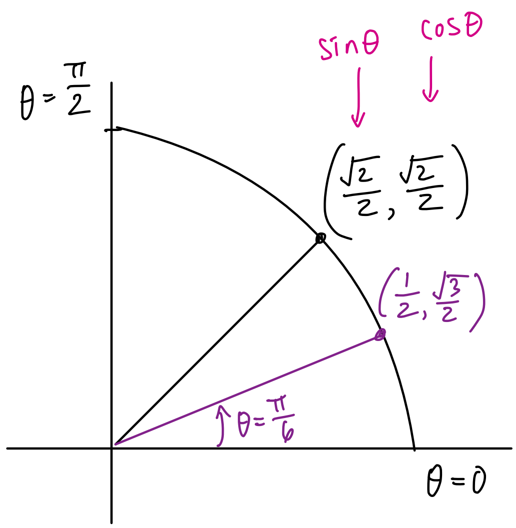
$$= \operatorname{Arctan}(u)\Big|_{\frac{1}{\sqrt{3}}}^{1}$$

$$= \operatorname{Arctan}(1) - \operatorname{Arctan}\left(\tfrac{1}{\sqrt{3}}\right)$$

$$= \frac{\pi}{4} - \frac{\pi}{6}$$

$$= \frac{3\pi - 2\pi}{12}$$

$$= \boxed{\frac{\pi}{12}}$$



How I got $\frac{\pi}{4}$ and $\frac{\pi}{6}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\tan\theta = 1 \Rightarrow \sin\theta = \cos\theta \Rightarrow \theta = \frac{\pi}{4}$$

$$\tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \sin\theta = \frac{1}{2},\ \cos\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{6}$$

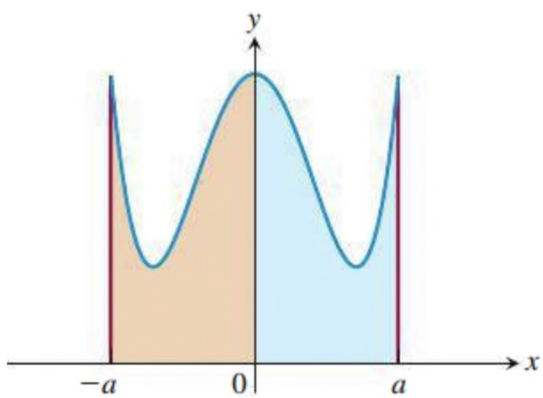
Thm:

If f is even (meaning $f(-x) = f(x)$) then

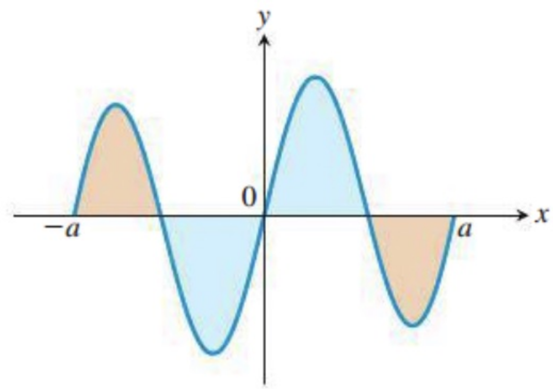
$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx$$

If f is odd (meaning $f(-x) = -f(x)$) then

$$\int_{-a}^{a} f(x)\,dx = 0$$



(a)



(b)

Ex: $\cos(x)$ is even.

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos(x)\,dx = 2\int_{0}^{\frac{\pi}{4}} \cos(x)\,dx$$

$$= 2\sin(x)\Big|_{0}^{\frac{\pi}{4}}$$

$$= 2\sin\left(\frac{\pi}{4}\right) - 2\sin(0)$$

$$= 2\,\frac{\sqrt{2}}{2} - 0$$

$$= \sqrt{2}$$

$\sin(x)$ is odd

$$\int_{-100}^{100} \sin(x)\,dx = 0$$

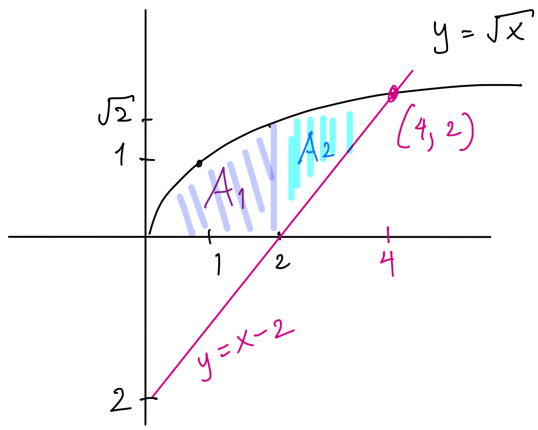
Def: If $f(x) \geq g(x)$ on $[a,b]$,

the area of the region between curves $y=f(x)$ and $y=g(x)$

from a to b is $\displaystyle\int_a^b \big(f(x) - g(x)\big)\, dx$

Like MML # 7

Ex Find the area of the region in the first quadrant that is bounded above by $y=\sqrt{x}$ and bounded below by the x-axis & line $y=x-2$.



To find intersection, set $\sqrt{x} = x-2$, solve:

$x = (x-2)^2$

$x = x^2 - 4x + 4$

$0 = x^2 - 5x + 4$

$0 = (x-1)(x-4)$

$x = 1, 4$

If $x=1$, $y = 1-2 = -1$

$(1,-1)$ is not in 1st Quadrant

If $x=4$, $y = 4-2 = 2$

$(4,2)$ is in 1st Quadrant

$$A_1 = \int_0^2 \sqrt{x}\, dx \qquad A_2 = \int_2^4 \Big(\sqrt{x} - (x-2)\Big)\, dx$$

Total area is $A_1 + A_2$

$$= \underbrace{\int_0^2 \sqrt{x}\, dx}_{\text{area of } A} + \underbrace{\int_2^4 (\sqrt{x} - x + 2)\, dx}_{\text{area of } B}$$

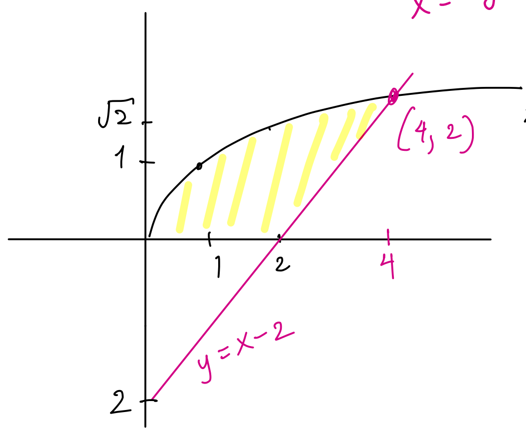
$$= \left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

$$= \frac{2}{3}(2)^{3/2} - 0 + \left(\frac{2}{3}(4)^{3/2} - 8 + 8\right) - \left(\frac{2}{3}(2)^{3/2} - 2 + 4\right)$$

$$= \frac{2}{3}(8) - 2 = \boxed{\frac{10}{3}.}$$

Alternative method (easier for this ex)

Integrate with respect to y



$x = y + 2$ is bigger function (more to the right)

$x = y^2$ is smaller function (more to the left)

$$\text{Area is } \int_{y=0}^{y=2} \left[(y+2) - y^2\right] dx = \frac{y^2}{2} + 2y - \frac{y^3}{3} \Bigg|_0^2$$

Now upper and lower bounds correspond to horizontal lines $y=0$ and $y=2$

$$= \frac{2^2}{2} + 2(2) - \frac{2^3}{3} - (0+0-0)$$

$$= 2 + 4 - \frac{8}{3}$$

$$= \frac{6+12-8}{3}$$

$$= \boxed{\frac{10}{3}}$$