Sec 5.5 Indefinite integrals and the substitution method

Recall Sec 4.8: $\int f(x)\,dx$ is the set of all antiderivatives of $f(x)$.

If we know one antiderivative $F(x)$ of $f(x)$, we can write

$$\int f(x)\,dx = F(x) + C$$

Def of <u>differential</u> (see Sec 3.11):

Let $u = g(x)$ be a differentiable function.

The <u>differential</u> dx is an independent variable.

The <u>differential</u> $du = g'(x)\,dx$ or $du = \left(\frac{du}{dx}\right) dx$

Ex: $u = x^3 + x \Rightarrow \frac{du}{dx} = 3x^2 + 1 \Rightarrow du = (3x^2+1)\,dx$

<u>Ex:</u> If $u(x)$ is differentiable and $n \neq -1$,

Ex: Say, $n = 22$

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) \underset{\text{Chain Rule}}{=} (n+1)\frac{u^n}{n+1} \cdot \frac{du}{dx} = u^n \frac{du}{dx}$$

This means $F(x) = \frac{u^{n+1}}{n+1}$ is an antiderivative of $f(x) = u^n \frac{du}{dx}$

So
$$\underbrace{\frac{u^{n+1}}{n+1} + C}_{F(x)+C} = \int \underbrace{\left(u^n \frac{du}{dx}\right)}_{f(x)} dx$$

Rewrite $\frac{du}{dx}\,dx$ as the differential du, so

$$\frac{u^{n+1}}{n+1} + C = \int u^n\,du$$

$$\boxed{\int u^n\,du = \frac{u^{n+1}}{n+1} + C}$$

Ex: Find the indefinite integral $\int (x^3+x)^5 \,(3x^2+1)\,dx$.

Answer: Set $u = x^3+x \Rightarrow \frac{du}{dx} = 3x^2+1 \Rightarrow \boxed{du = (3x^2+1)\,dx}$

may skip middle step

$$\int (x^3+x)^5 \boxed{(3x^2+1)\,dx} = \int u^5 \boxed{du}$$

$$= \frac{u^6}{6} + C$$

$$= \boxed{\frac{(x^3+x)^6}{6} + C}$$

Ex: Evaluate the indefinite integral $\int \sqrt{2x+1}\,dx$

Answer: Try $u = 2x+1 \Rightarrow \frac{du}{dx} = 2 \Rightarrow du = 2\,dx$

$\Rightarrow \frac{1}{2}du = dx$

$$\int \sqrt{2x+1}\,\boxed{dx} = \int \sqrt{u}\,\boxed{\tfrac{1}{2}du}$$

$$= \frac{1}{2}\int u^{\frac{1}{2}}\,du$$

if $n \neq -1$

$$= \frac{1}{2}\,\frac{u^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + C \qquad \left(\text{because } \int u^n\,du = \frac{u^{n+1}}{n+1} + C\right)$$

$$= \frac{1}{2}\,\frac{2}{3}\,u^{\frac{3}{2}} + C$$

$$= \boxed{\frac{1}{3}(2x+1)^{\frac{3}{2}} + C} \qquad (\text{Sub back } u = 2x+1)$$

Thm (The substitution Rule or "u-substitution")

If $u(x)$ is differentiable and f is continuous, then

$$\int f(u(x)) \cdot u'(x)\, dx = \int f(u)\, du$$

<u>Why?</u>

Let F be an antiderivative of f,

that is, $F' = f$

$$\frac{d}{dx} F(u(x)) = \frac{dF}{du} \cdot \frac{du}{dx} \quad \text{by Chain Rule}$$

$$= \boxed{f(u(x)) \cdot \frac{du}{dx}}$$

$$\int f(u)\, du = \int F'(u)\, du \quad \text{since } F' = f$$

$$= F(u) + C$$

(main thm in Sec 4.8 says the most general antiderivative of f is $F(u) + C$)

$$= F(u(x)) + C$$

$$= \int \left(\frac{d}{dx} F(u(x)) \right) dx$$

$$= \int \boxed{\left(f(u(x)) \cdot \frac{du}{dx} \right)} dx$$

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EXAMPLE 3 Find $\int \sec^2(5x+1)\cdot 5\,dx$.

Solution We substitute $u = 5x + 1$ and $du = 5\,dx$. Then

$$\int \sec^2(5x+1)\cdot 5\,dx = \int \sec^2 u\,du \qquad \text{Let } u = 5x+1,\ du = 5\,dx.$$

$$= \tan u + C \qquad \frac{d}{du}\tan u = \sec^2 u$$

$$= \tan(5x+1) + C. \qquad \text{Substitute } 5x+1 \text{ for } u.$$

EXAMPLE 4 Find $\int \cos(7\theta + 3)\,d\theta$.

Solution We let $u = 7\theta + 3$ so that $du = 7\,d\theta$. There is a factor of 7 in this formula for du, but there is no corresponding 7 preceding $d\theta$ in the integral. We can compensate for this by multiplying and dividing by 7, using the same procedure as in Example 2. Then

$$\int \cos(7\theta+3)\,d\theta = \frac{1}{7}\int \cos(7\theta+3)\cdot 7\,d\theta \qquad \text{Place factor } 1/7 \text{ in front of integral.}$$

$$= \frac{1}{7}\int \cos u\,du \qquad \text{Substitute } u = 7\theta+3,\ du = 7\,d\theta.$$

$$= \frac{1}{7}\sin u + C \qquad \text{Integrate.}$$

$$= \frac{1}{7}\sin(7\theta+3) + C. \qquad \text{Replace } u \text{ by } 7\theta+3.$$

Ex: $$\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx = \int \frac{-du}{u} \qquad u = \cos x,\ du = -\sin x\,dx$$

$$= -\ln|u| + C = -\ln|\cos x| + C$$

$$= \ln\frac{1}{|\cos x|} + C = \ln|\sec x| + C \qquad \text{Reciprocal Rule}$$

Ex: $$\int \sec x\,dx = \int (\sec x)(1)\,dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}dx \qquad \frac{\sec x + \tan x}{\sec x + \tan x} \text{ is equal to 1.}$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}dx$$

$$= \int \frac{du}{u} \qquad u = \tan x + \sec x,\ du = (\sec^2 x + \sec x \tan x)\,dx$$

$$= \ln|u| + C = \ln|\sec x + \tan x| + C.$$

Integrals of the tangent, cotangent, secant, and cosecant functions

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int \cot x\,dx = \ln|\sin x| + C \qquad \int \csc x\,dx = -\ln|\csc x + \cot x| + C$$

Ex:

$$\int \frac{x}{\sqrt{x-10}}\,dx = \int \frac{u+10}{\sqrt{u}}\,du = \int \left(\frac{u}{\sqrt{u}} + \frac{10}{\sqrt{u}}\right) du$$

$$\boxed{u = x-10} \Rightarrow u+10 = x$$

$$du = 1\,dx$$

$$= \int \left(u^{\frac{1}{2}} + 10\,u^{-\frac{1}{2}}\right) du$$

$$= \frac{u^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + 10\,\frac{u^{-\frac{1}{2}+1}}{\left(-\frac{1}{2}+1\right)} + C$$

$$= \frac{2}{3}u^{\frac{3}{2}} + 10\,\frac{u^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + C$$

$$= \frac{2}{3}u^{\frac{3}{2}} + 20\,u^{\frac{1}{2}} + C$$

$$= \boxed{\frac{2}{3}(x-10)^{\frac{3}{2}} + 20\,(x-10)^{\frac{1}{2}} + C}$$

(A very similar example)

EXAMPLE 6 Evaluate $\int x\sqrt{2x+1}\,dx$.

Solution Our previous experience with the integral in Example 2 suggests the substitution $u = 2x + 1$ with $du = 2\,dx$. Then

$$\sqrt{2x+1}\,dx = \frac{1}{2}\sqrt{u}\,du.$$

However, in this example the integrand contains an extra factor of x that multiplies the factor $\sqrt{2x+1}$. To adjust for this, we solve the substitution equation $u = 2x + 1$ for x to obtain $x = (u-1)/2$ and find that

$$x\sqrt{2x+1}\,dx = \frac{1}{2}(u-1)\cdot\frac{1}{2}\sqrt{u}\,du.$$

The integration now becomes

$$\int x\sqrt{2x+1}\,dx = \frac{1}{4}\int (u-1)\sqrt{u}\,du = \frac{1}{4}\int (u-1)u^{1/2}\,du \qquad \text{Substitute.}$$

$$= \frac{1}{4}\int \left(u^{3/2} - u^{1/2}\right) du \qquad \text{Multiply terms by } u^{1/2}.$$

$$= \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C \qquad \text{Integrate.}$$

$$= \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C. \qquad \text{Replace } u \text{ by } 2x+1. \ \blacksquare$$

EXAMPLE 9 Evaluate $\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}$.

Solution We will use the substitution method of integration as an exploratory tool: We substitute for the most troublesome part of the integrand and see how things work out. For the integral here, we might try $u = z^2 + 1$, or we might even press our luck and take u to be the entire cube root. In this example both substitutions turn out to be successful, but that is not always the case. If one substitution does not help, a different substitution may work instead.

Method 1: Substitute $u = z^2 + 1$.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{du}{u^{1/3}} \qquad \text{Let } u = z^2 + 1,\ du = 2z\,dz.$$

$$= \int u^{-1/3}\,du \qquad \text{In the form } \int u^n\,du$$

$$= \frac{u^{2/3}}{2/3} + C \qquad \text{Integrate.}$$

$$= \frac{3}{2}u^{2/3} + C$$

$$= \frac{3}{2}(z^2+1)^{2/3} + C \qquad \text{Replace } u \text{ by } z^2 + 1.$$

Method 2: Substitute $u = \sqrt[3]{z^2+1}$ instead.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{3u^2\,du}{u} \qquad \text{Let } u = \sqrt[3]{z^2+1},\ u^3 = z^2 + 1,\ 3u^2\,du = 2z\,dz.$$

$$= 3\int u\,du$$

$$= 3 \cdot \frac{u^2}{2} + C \qquad \text{Integrate.}$$

$$= \frac{3}{2}(z^2+1)^{2/3} + C \qquad \text{Replace } u \text{ by } (z^2+1)^{1/3}.$$

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