Sec 5.4 The Fundamental Theorem of Calculus (FTC)

Developed by Leibniz & Newton (independent of each other) during 1600s.

Mean Value Theorem If f is continuous on $[a,b]$, then there is c in $[a,b]$ such that

$$f(c) = \underbrace{\frac{1}{b-a} \overbrace{\int_a^b f(x)\,dx}^{\text{Defined as limit of Riemann sums}}}_{\text{The average value of } f \text{ on } [a,b]}$$

(If $f(x) \geq 0$, also defined as area under the curve $y=f(x)$ over $[a,b]$)

In other words, there is c in $[a,b]$ such that $f(c)$ equals the average value of f on $[a,b]$.

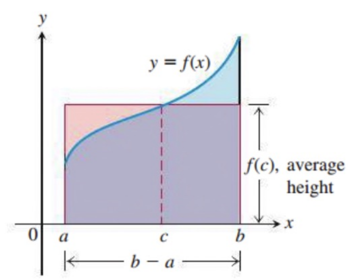


FIGURE 5.16 The value $f(c)$ in the Mean Value Theorem is, in a sense, the average (or *mean*) height of f on $[a,b]$. When $f \geq 0$, the area of the rectangle is the area under the graph of f from a to b,

$$f(c)(b-a) = \int_a^b f(x)\,dx.$$

Ex:

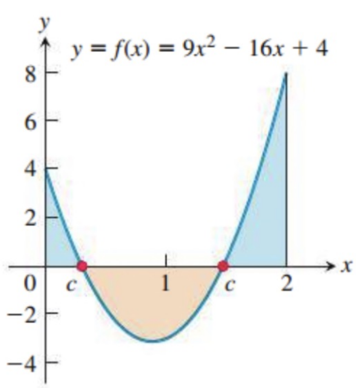


FIGURE 5.18 The function $f(x) = 9x^2 - 16x + 4$ satisfies $\int_0^2 f(x)\,dx = 0$, and there are two values of c in the interval $[0, 2]$ where $f(c) = 0$.

Fundamental Thm Part 1

an open interval

Suppose f is continuous on I, and let $a \in I$.

If $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right]$$

Ex: Use the Fundamental Thm Part 1 to find $\frac{dy}{dx}$

for $y = \int_4^x \frac{1}{\ln t}\,dt$.

Answer: Let $F(x) = \int_4^x \frac{1}{\ln t}\,dt$

$f(t)$

$$F'(x) = \frac{d}{dx}\left[\int_4^x \frac{1}{\ln t}\,dt\right] = f(x) = \frac{1}{\ln(x)}$$

FTC Part 1

$$\boxed{\frac{dy}{dx} = \frac{1}{\ln x}}$$

More challenging Ex:

Use the Fundamental Thm Part 1 to find $\frac{dy}{dx}$

for $y = \int_1^{x^2} \cos(t)\, dt$

Answer:

note the upper limit of integration is not x

- $y = \int_1^{x^2} \cos(t)\, dt$ is a composition of two functions

$F(u) = \int_1^{u} \cos(t)\, dt$ and $u(x) = x^2$:

$y(x) = F(u(x)) = F(x^2)$

- $\frac{dF}{du} = \frac{d}{du}\left[\int_1^{u} \cos(t)\, dt\right] = \cos(u)$

FTC Part 1

- $\frac{du}{dx} = \frac{d}{dx}(x^2) = 2x$

Chain Rule

- $\frac{dy}{dx} = \frac{dF}{du} \cdot \frac{du}{dx}$

$= \cos(u) \cdot 2x$

$= \boxed{\cos(x^2)\ 2x}$

In general,

$$\boxed{\frac{d}{dx}\left[\int_a^{u(x)} f(t)\, dt\right] = f(u(x)) \cdot \frac{du}{dx}}$$

Similar Ex:

Use the Fundamental Thm Part 1 to find $\frac{dy}{dx}$

for $y = \int_1^{\sqrt{x}} \cos(t)\, dt$

Answer:

note the upper limit of integration is not x

- $y = \int_1^{\sqrt{x}} \cos(t)\, dt$ is a composition of two functions

$F(u) = \int_1^{u} \cos(t)\, dt$ and $u(x) = \sqrt{x}$:

$y(x) = F(u(x)) = F(\sqrt{x})$

- $\frac{dF}{du} = \frac{d}{du}\left[\int_1^{u} \cos(t)\, dt\right] = \cos(u)$

FTC Part 1

- $\frac{du}{dx} = \frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2} \frac{1}{x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$

Chain Rule

- $\frac{dy}{dx} = \frac{dF}{du} \cdot \frac{du}{dx}$

$= \cos(u) \cdot \frac{1}{2\sqrt{x}}$

$= \boxed{\frac{\cos(\sqrt{x})}{2\sqrt{x}}}$

Another Ex (for FTC Part 1):

Find $\frac{dy}{dx}$ for $y = \int_{\tan(x)}^{0} \frac{1}{1+t^2}\,dt$.

Answer:

$$\int_{\tan(x)}^{0} \frac{1}{1+t^2}\,dt = -\int_{0}^{\tan(x)} \frac{1}{1+t^2}\,dt$$

Rule 1 of definite integrals from Sec 5.3

$$\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[-\int_0^{\tan(x)} \frac{1}{1+t^2}\,dt\right]$$

$$= -\frac{d}{dx}\left[\int_0^{\tan(x)} \frac{1}{1+t^2}\,dt\right]$$

Here $f(u) = \frac{1}{1+u^2}$

$u(x) = \tan(x)$

$$= -\frac{1}{1+u^2}\cdot\frac{du}{dx}$$

$$= -\frac{1}{1+(\tan(x))^2}\cdot\frac{d}{dx}(\tan(x))$$

$$= -\frac{(\sec(x))^2}{1+(\tan(x))^2}$$

Trig identities:

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \Rightarrow \tan^2 x + 1 = \sec^2 x$$

$$= -\frac{\sec^2(x)}{\sec^2(x)}$$

$$= \boxed{-1}$$

Fundamental Thm Part 2

Suppose f is continuous on $[a, b]$.

If F is an antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

<u>Notation:</u> write $F(x)\Big|_a^b$ or $F(x)\Big]_a^b$ or $\Big[F(x)\Big]_a^b$ to mean $F(b) - F(a)$.

<u>Remark:</u> FTC Part 2 gives us a connection between

definite integral $\int_a^b f(x)\, dx$ (limit of Riemann sums) and

indefinite integral $\int f(x)\, dx$ (functions whose derivative is $f(x)$):

$$\underbrace{\int_a^b f(x)\, dx}_{\text{definite integral}} = \underbrace{F(x)}_{\text{any antiderivative (indefinite integral)}}\Big]_a^b$$

<u>Ex</u> (for FTC Part 2):

* $$\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx = \left[x^{3/2} + \frac{4}{x}\right]_1^4$$
$$= \left[(4)^{3/2} + \frac{4}{4}\right] - \left[(1)^{3/2} + \frac{4}{1}\right]$$
$$= [8 + 1] - [5] = 4$$

$$\frac{d}{dx}\left(x^{3/2} + \frac{4}{x}\right) = \frac{3}{2}x^{1/2} - \frac{4}{x^2}$$

* $$\int_0^1 \frac{dx}{x+1} = \ln|x+1|\Big]_0^1$$
$$= \ln 2 - \ln 1 = \ln 2$$

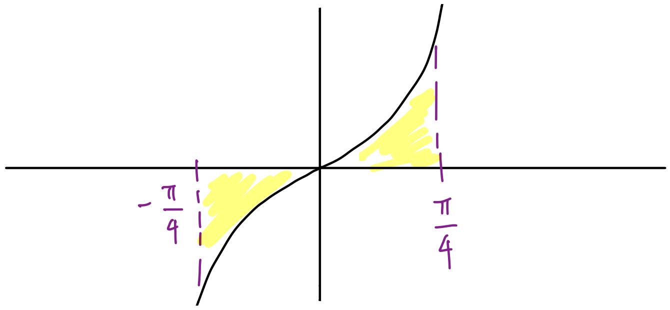
$$\frac{d}{dx}\ln|x+1| = \frac{1}{x+1}$$

* $$\int_0^1 \frac{dx}{x^2+1} = \tan^{-1} x\Big]_0^1$$
$$= \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

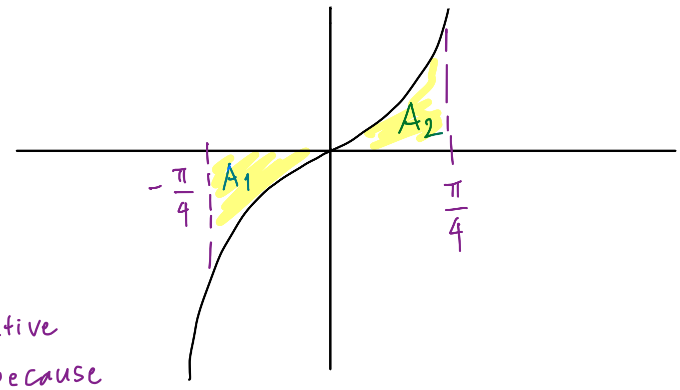
$$\frac{d}{dx}\tan^{-1} x = \ = \frac{1}{x^2+1}$$

More Ex (for FTC Part 2):

Find the total area between the region $y = \sec(x)\tan(x)$ and the x-axis, between $x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$.



Answer:



We put negative sign here because $\sec x \tan x$ is below x-axis on $[-\frac{\pi}{4}, 0]$ but we want A_1 to be positive

because $\frac{d}{dx} \sec x = \sec x \tan x$

So $\sec x$ is an antiderivative of $\sec x \tan x$

$$A_1 = -\int_{-\frac{\pi}{4}}^{0} \sec(x)\tan(x)\,dx = -\Big[\sec x\Big]_{-\frac{\pi}{4}}^{0}$$

$$= -\left[\sec(0) - \sec\left(-\frac{\pi}{4}\right)\right]$$

$$= -\left[\frac{1}{\cos(0)} - \frac{1}{\cos\left(-\frac{\pi}{4}\right)}\right]$$

$\sec(t) = \frac{1}{\cos(t)}$

$$= -\left[1 - \frac{1}{\left(\frac{\sqrt{2}}{2}\right)}\right]$$

$$= -1 + \frac{2}{\sqrt{2}} = -1 + \sqrt{2}$$

$$A_2 = \int_0^{\frac{\pi}{4}} \sec x \tan x\,dx = \sec x\Big]_0^{\frac{\pi}{4}} = \sec\frac{\pi}{4} - \sec(0) = \sqrt{2} - 1$$

$$\text{Total area} = A_1 + A_2 = (\sqrt{2} - 1) + (\sqrt{2} - 1) = \boxed{2\sqrt{2} - 2}$$

Displacement & Distance Traveled Ex:

- A rock is blown straight up from the ground.

Velocity of the rock after t secs is $v(t) = 160 - 32t$ ft/sec.

- Position of the rock after t secs is

$$\int_0^t v(t)\,dt = \int_0^t 160 - 32t \quad \text{ft}$$

- Total distance traveled after t secs is

$$\int_0^t |v(t)|\,dt = \int_0^t |160 - 32| \quad \text{ft}$$

when the rock moves in the negative direction, we want to think of it as moving in the positive direction

- Q: Find the position of the rock after 8 secs

Ans: $$\int_0^8 (160 - 32t)\,dt = \left[160t - \frac{32t^2}{2}\right]_0^8$$

$$= 160(8) - \frac{32}{2}(8) - [0 - 0]$$

$$= 256 \text{ ft}$$

- Q: Find total distance traveled after 8 secs.

Ans: $$\int_0^8 |v(t)|\,dt = \int_0^5 |v(t)|\,dt + \int_5^8 |v(t)|\,dt$$

$$= \int_0^5 (160 - 32t)\,dt - \int_5^8 (160 - 32t)\,dt \qquad |v(t)| = -(160 - 32t) \text{ over } [5, 8]$$

$$= \Big[160t - 16t^2\Big]_0^5 - \Big[160t - 16t^2\Big]_5^8$$

$$= [(160)(5) - (16)(25)] - [(160)(8) - (16)(64) - ((160)(5) - (16)(25))]$$

$$= 400 - (-144) = 544. \text{ ft}$$