Recall from Sec 5.2

Riemann Sums

- A partition of an interval $[a,b]$ is a set of points $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ with $x_0 = a$ and $x_n = b$, dividing $[a,b]$ into n closed subintervals

$$[x_0, x_1],\ [x_1, x_2],\ \ldots,\ [x_{n-1}, x_n].$$

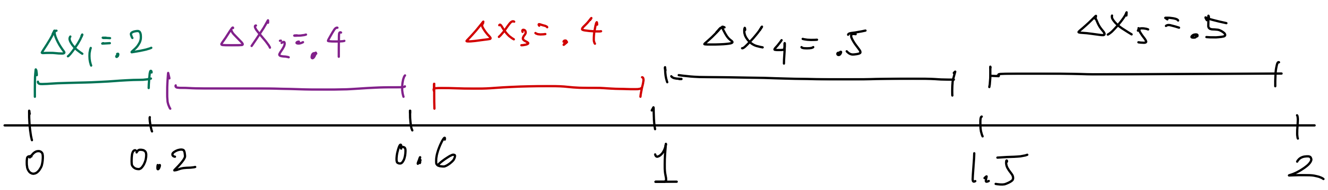
- The lengths of the subintervals are denoted $\Delta x_1,\ \Delta x_2,\ \ldots,\ \Delta_{n-1}$

  If all subintervals have equal width, their width is $\frac{b-a}{n}$.

  Call this common width Δx.

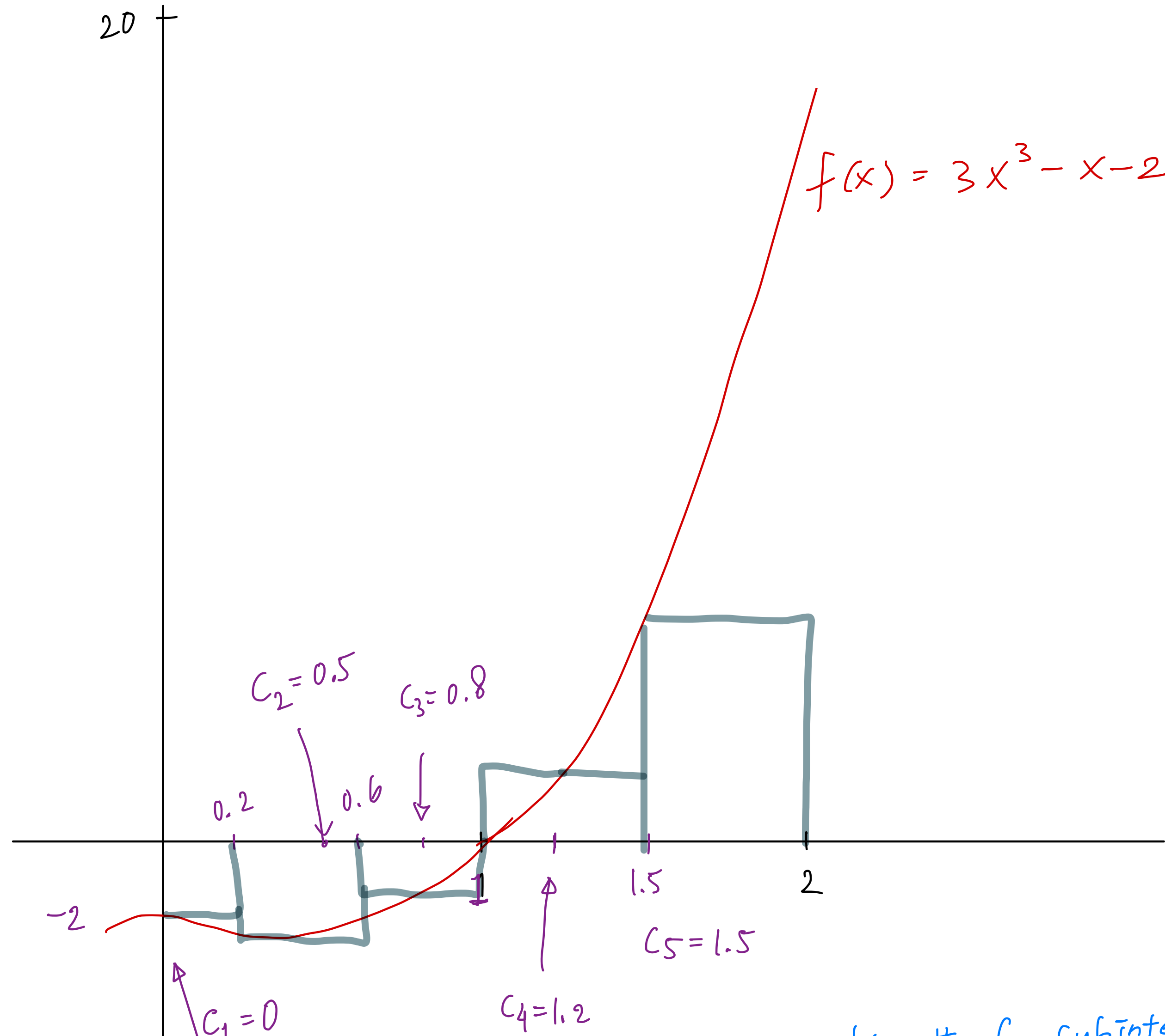
- The norm $\|P\|$ of P is the largest of all the subinterval widths.

Ex: $P = \{0, .2, .6, 1, 1.5, 2\}$ is a partition of $[0,2]$.



Norm of P is $\|P\| = 0.5$.

- For each subinterval $[x_{k-1}, x_k]$:
  - we select some point c_k in $[x_{k-1}, x_k]$
  - draw a rectangle from x-axis to $f(c_k)$
  - Area of rectangle is $f(c_k)\,\Delta_k$



this is the # of subintervals of the partition

- The sum $\displaystyle\sum_{k=1}^{5} f(c_k)\,\Delta x_k =$

$f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + f(c_3)\Delta x_3 + f(c_4)\Delta x_4 + f(c_5)\Delta x_5 =$

$f(0)\,0.2 + f(0.5)\,0.4 + f(0.8)\,0.4 + f(1.2)\,0.5 + f(1.5)\,0.5$

is an example of a Riemann sum for f on the interval $[a,b] = [0,2]$.

Sec 5.3 The Definite integral

- Let the norm $\|P\|$ approach 0.
- If the Riemann sums of f on $[a,b]$ approach a number, this number is called the definite integral of f over $[a,b]$.
- If $\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k) \Delta x_k$ exists and equal a number J,

no matter what choices we make for partition P of $[a,b]$

the number J is called the definite integral of f from a to b,

upper limit of integration

integrand

x is variable of integration

denoted $\int_a^b f(x)\, dx$ exists

lower limit of integration

We also say: ① " the Riemann sums of f on $[a,b]$ converge to J "

② "f is integrable over $[a,b]$."

- Note: The definite integral depends on the f, not on our choice of letter.

Ex: $\int_a^b f(t)\, dt$ is the same as $\int_a^b f(u)\, du$.

- If not all Riemann sums for f converge to the same number J, we say f is <u>not integrable</u>.

- Thm: If f is continuous over $[a,b]$, then f is integrable over $[a,b]$ (i.e. the definite integral $\int_a^b f(x)\,dx$ exists)

Assume functions f, g are integrable over $[a,b]$.

1.) $\int_b^a f(x)\,dx \overset{\text{Def}}{=} -\int_a^b f(x)\,dx$ (when you interchange a,b)

2.) $\int_a^a f(x)\,dx \overset{\text{Def}}{=} 0$

(The rest of the rules are theorems)

3.) $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$ for any number k

"Constant Multiple Rule"

4.) $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

"Sum Rule"

5.) $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$ if $a \le c \le b$

6.) $\begin{pmatrix} \text{minimum} \\ \text{value of } f(x) \\ \text{on } [a,b] \end{pmatrix} \cdot (b-a) \le \int_a^b f(x)\,dx \le \begin{pmatrix} \text{maximum} \\ \text{value of } f(x) \\ \text{on } [a,b] \end{pmatrix} \cdot (b-a)$

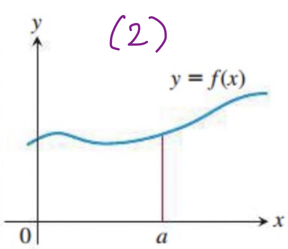
7.) If $f(x) \geq g(x)$ for all x in $[a,b]$,

then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

Special case of Rule 6:

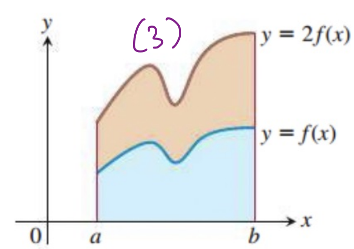
If $f(x) \geq 0$ for all x in $[a,b]$,

then $\int_a^b f(x)\,dx \geq 0$.



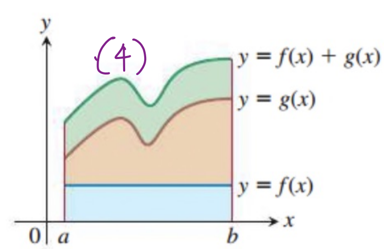
(a) *Zero Width Interval:*

$$\int_a^a f(x)\,dx = 0$$



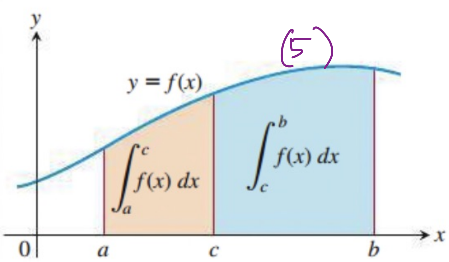
(b) *Constant Multiple:* $(k = 2)$

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$



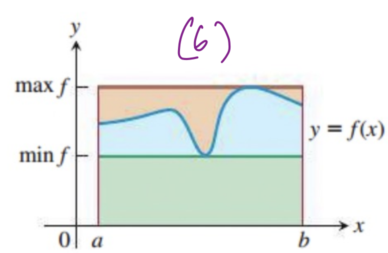
(c) *Sum: (areas add)*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$



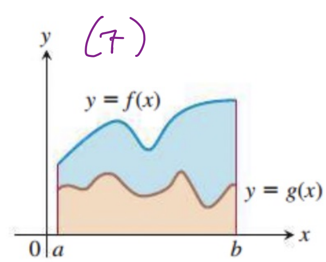
(d) *Additivity for Definite Integrals:*

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$



(e) *Max-Min Inequality:*

$$(\min f) \cdot (b - a) \le \int_a^b f(x)\,dx \le (\max f) \cdot (b - a)$$



(f) *Domination:*

If $f(x) \geq g(x)$ on $[a, b]$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

FIGURE 5.11 Geometric interpretations of Rules 2–7 in Table 5.6.

Like MML #2, 3, 4

EXAMPLE 2 To illustrate some of the rules, we suppose that

$$\int_{-1}^{1} f(x)\,dx = 5, \qquad \int_{1}^{4} f(x)\,dx = -2, \quad \text{and} \quad \int_{-1}^{1} h(x)\,dx = 7.$$

Then

1. $$\int_{4}^{1} f(x)\,dx = -\int_{1}^{4} f(x)\,dx = -(-2) = 2$$ Rule 1

2. $$\int_{-1}^{1} [2f(x) + 3h(x)]\,dx = 2\int_{-1}^{1} f(x)\,dx + 3\int_{-1}^{1} h(x)\,dx$$ Rules 3 and 4

$$= 2(5) + 3(7) = 31$$

3. $$\int_{-1}^{4} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx = 5 + (-2) = 3$$ Rule 5

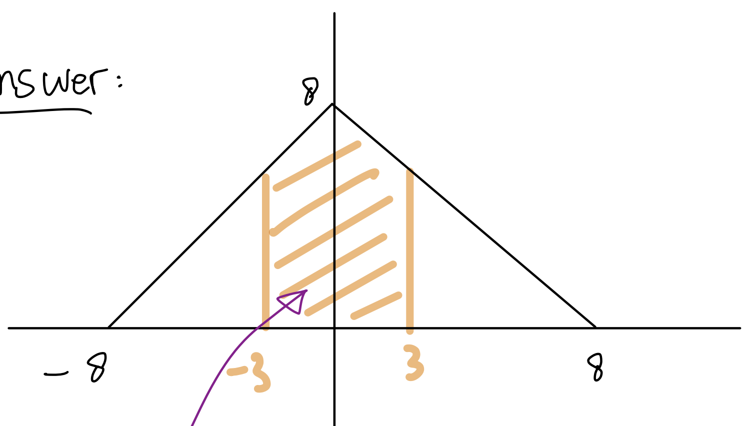
Def If $f(x) \geq 0$ and integrable over $[a,b]$, then define the area under the curve $y = f(x)$ over $[a,b]$ to be the definite integral of f from a to b

$$\text{Area} = \int_a^b f(x)\,dx$$

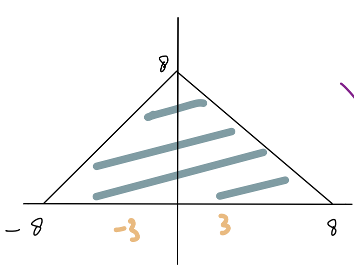
Ex: Consider the definite integral $\int_{-3}^{3} \underbrace{(8 - |x|)}_{\text{the integrand } f(x) = 8 - |x|} \, dx$.

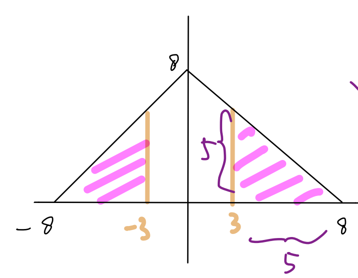
Graph $y = f(x)$ and use area formulas to evaluate the integral

Answer:



This area is

(area of 

) − (area of 

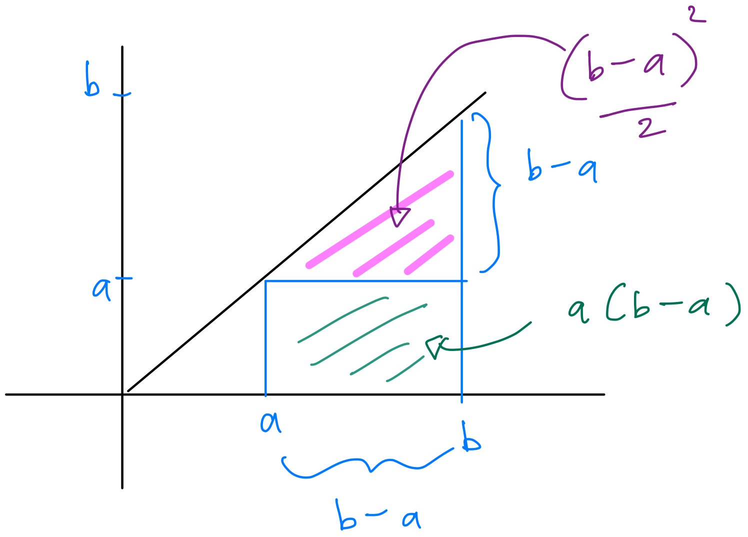
)

$= 8 \cdot 8 - 5 \cdot 5$

$= 64 - 25$

$= \boxed{39}$

Ex: $\int_a^b x \, dx = \boxed{?}$ if $a < b$



Total area is $\frac{(b-a)^2}{2} + a(b-a)$

$$= \frac{b^2 + a^2 - 2ab}{2} + ab - a^2$$

$$= \frac{b^2}{2} + \frac{a^2}{2} - \cancel{ab} + \cancel{ab} - a^2 \frac{2}{2}$$

$$= \boxed{\frac{b^2}{2} - \frac{a^2}{2}}$$

Fact:

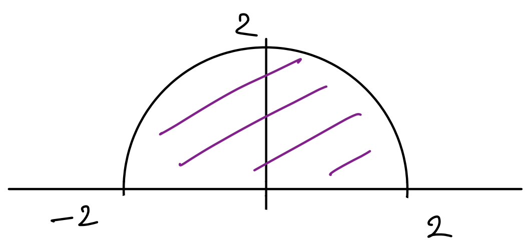
$$\boxed{\int_a^b x \, dx = \frac{b^2}{2} - \frac{a^2}{2} \quad \text{if } a < b}$$

Ex: Graph the integrand & use areas to evaluate the definite integral

$$\int_{-2}^{2} \sqrt{4-x^2}\, dx$$

Answer: $y^2 + x^2 = 2^2$ is circle w/ radius 2 centered at the origin.

$y = \sqrt{4-x^2}$ is the upper semicircle



Area of the circle is $\pi\,(\text{radius})^2 = \pi\,(2)^2 = 4\pi$

Area of half the circle is $\boxed{2\pi}$.