5.2 $\underbrace{\text{Sigma}}_{\text{Greek letter } \Sigma}$ notation & limits of finite sums

Sum in sigma notation	meaning
$\sum_{k=1}^{n} a_k$	$a_1 + a_2 + a_3 + \dots + a_n$
$\sum_{k=1}^{6} k^2$	$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2$
$\sum_{k=2}^{5} (-1)^k\, k$	$(-1)^2\, 2 + (-1)^3\, 3 + (-1)^4\, 4 + (-1)^5\, 5 = 2 - 3 + 4 - 5$
$\sum_{j=4}^{5} \frac{j^2}{j-1}$	$\frac{4^2}{4-1} + \frac{5^2}{5-1} = \frac{16}{3} + \frac{25}{4} = \frac{139}{12}$
$\sum_{i=-1}^{2} 7\, f(i)$	$7f(-1) + 7f(0) + 7f(1) + 7f(2) = 7\left(f(-1) + f(0) + f(1) + f(2)\right)$

$$= 7 \sum_{i=-1}^{2} 7\, f(i)$$

If $f(x) = x^3$

$$= 7\left((-1)^3 + 0^3 + 1^3 + 2^3\right)$$

$$= 7(-1 + 1 + 2)$$

$$= 14$$

Ex: Express the sum $1 + 3 + 5 + 7 + 9$ in sigma notation.

Use 2 as the lower limit of summation and

k for the index of summation.

Answer: Look for a pattern.

Observe that to go to the next term we always add $+2$

a_2	+	a_3	+	a_4	+	a_5	+	a_6
1		$(1+2)$		$(1+2+2)$		$(1+2+2+2)$		$(1+2+2+2+2)$
$1+2(0)$		$1+2(1)$		$1+2(2)$		$1+2(3)$		$1+2(4)$
$1+2(2-2)$		$1+2(3-2)$		$1+2(4-2)$		$1+2(5-2)$		$1+2(6-2)$
$\uparrow$ k		$\uparrow$ k		$\uparrow$ k		$\uparrow$ k		$\uparrow$ k

$$\sum_{k=2}^{6} 1 + 2(k-2) \quad \text{or} \quad \sum_{k=2}^{6} 1 + 2k - 4 \quad \text{or} \quad \boxed{\sum_{k=2}^{6} 2k - 3}$$

Ex: Express the sum $-\frac{5}{3} + \frac{5}{9} - \frac{5}{27} + \frac{5}{81}$ in sigma notation.

Use 2 as the lower limit of summation and j for the index of summation.

Answer Look for a pattern.

$$a_2 \quad + \quad a_3 \quad + \quad a_4 \quad + \quad a_5$$

$(-1)^{odd}$ $\quad$ $(-1)^{even}$ $\quad$ $(-1)^{odd}$ $\quad$ $(-1)^{even}$

$$-\frac{5}{3^{1}} + \frac{5}{3^{2}} - \frac{5}{3^{3}} + \frac{5}{3^{4}} = \sum_{j=2}^{5} (-1)^{j+1} \frac{5}{3^{j-1}} \quad \text{or} \quad \sum_{j=2}^{5} (-1)^{j+1}\, 3\left(\frac{5}{3^{j}}\right)$$

$2-1$ $\quad$ $3-1$ $\quad$ $4-1$ $\quad$ $5-1$

Ex: Express $-\frac{1}{26} + \frac{1}{27} - \frac{1}{28} + \frac{1}{29} - \frac{1}{30}$ in sigma notation.

Use 25 as the lower limit and i for the index.

Algebra Rules for finite sums

$$\sum_{k=1}^{3} (k + k^2) = (1 + 1^2) + (2 + 2^2) + (3 + 3^2)$$

$$= (1 + 2 + 3) + (1^2 + 2^2 + 3^2) \qquad \text{Regroup terms}$$

$$= \sum_{k=1}^{3} k + \sum_{k=1}^{3} k^2$$

In general,

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \qquad \text{Sum Rule}$$

$$\sum_{k=1}^{n} c\, a_k = c \sum_{k=1}^{n} a_k \qquad \text{Constant Multiple Rule}$$

EX: If $\sum_{k=2}^{100} a_k = 4$ and $\sum_{k=2}^{100} b_k = 24$, find ...

1.) $$\sum_{k=1}^{100} \frac{b_k}{24} = \frac{1}{24} \sum_{k=1}^{100} b_k$$

$$= \frac{1}{24}(24)$$

$$= 1$$

2.) $$\sum_{k=2}^{100} b_k - 3a_k = \sum_{k=2}^{100} b_k + (-3)\sum_{k=2}^{100} a_k$$

$$= 24 + (-3)4$$

$$= 12$$

Riemann Sums

- A <u>partition</u> of an interval $[a, b]$ is a set of points $x_0, x_1, x_2, \dots, x_{n-1}, x_n$ (with $x_0 = a$ and $x_n = b$) dividing $[a, b]$ into n closed subintervals

$$[x_0, x_1], \ [x_1, x_2], \ \dots, \ [x_{n-1}, x_n].$$

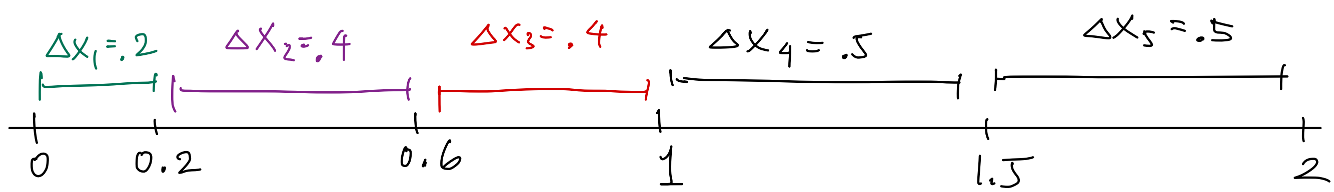
- The lengths of the subintervals are denoted $\Delta x_1, \ \Delta x_2, \ \dots, \ \Delta_{n-1}$

  If all subintervals have equal width, their width is $\frac{b-a}{n}$.

  Call this common width Δx.

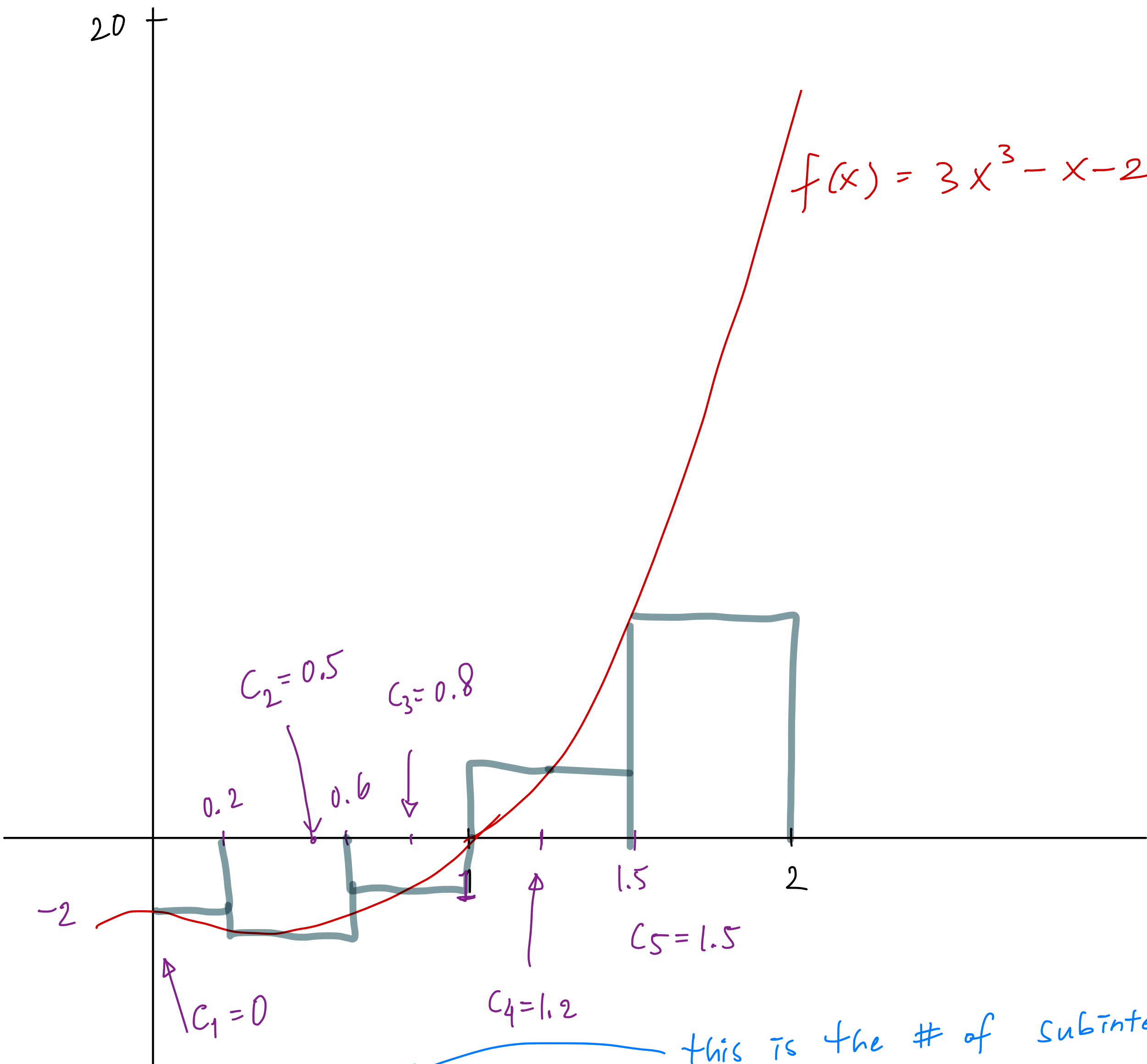
- The <u>norm</u> $\|P\|$ of P is the largest of all the subinterval widths.

Ex: $P = \{0, .2, .6, 1, 1.5, 2\}$ is a partition of $[0, 2]$.



Norm of P is $\|P\| = 0.5$.

- For each subinterval $[x_{k-1}, x_k]$:
  - we select some point c_k in $[x_{k-1}, x_k]$
  - draw a rectangle from x-axis to $f(c_k)$
  - Area of rectangle is $f(c_k)\, \Delta_k$



this is the # of subintervals of the partition

- The sum $\sum_{k=1}^{5} f(c_k)\, \Delta x_k =$

$f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + f(c_3)\Delta x_3 + f(c_4)\Delta x_4 + f(c_5)\Delta x_5 =$

$f(0)\, 0.2 + f(0.5)\, 0.4 + f(0.8)\, 0.4 + f(1.2)\, 0.5 + f(1.5)\, 0.5$

is an example of a Riemann sum for f on the interval $[a,b] = [0,2]$.

Next time (Sec 5.3 The Definite integral):

- Let the norm $\|P\|$ approach 0.
- If the Riemann sums of f on $[a,b]$ approach a number, this number is called the <u>definite integral of f over $[a,b]$</u>.