Ch 5 Integrals Big Ideas:

There are simple formulas for calculating lengths, areas & volumes of straight line segments, triangles, spheres, cones.

For more general curves & shapes, we use a powerful tool called the <u>definite integral</u> (defined to be a limit of increasingly fine approximations).

Sec 5.1 Area & Estimating with Finite sums

Examples of applications:

Approximate
1. area under a curve
2. distance traveled
3. average value of a function

1 Area

Ex: Consider the region R

below $y = 1 - x^2$,

above the x-axis, &

between $x = 0$ and $x = 1$

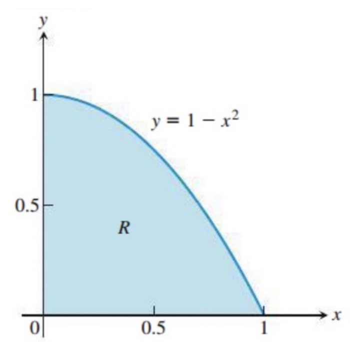
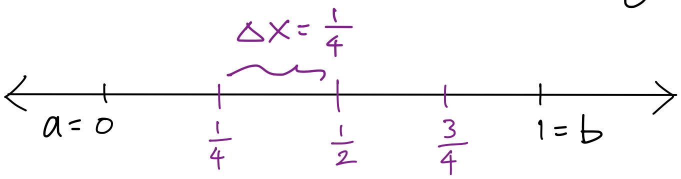
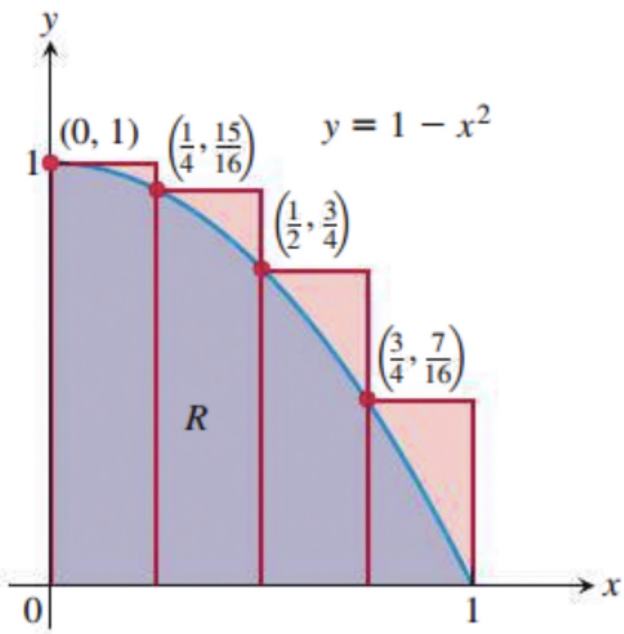


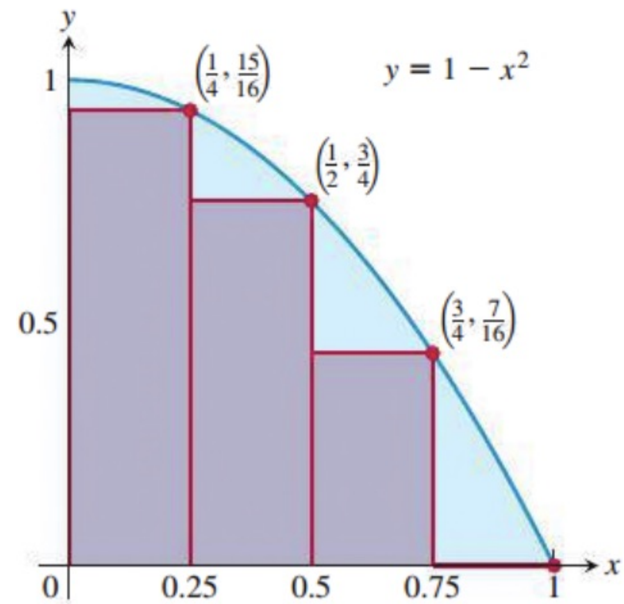
FIGURE 5.1 The area of the shaded region R cannot be found by a simple formula.

- There is no simple formula for computing the area of R, but we can estimate it using areas of rectangles.
- In general, the region R is below $y = f(x)$, above the x-axis, & between the vertical lines $x = a$ and $x = b$.

Partition / subdivide the interval $[a,b] = [0,1]$ into, say, $n = 4$ subintervals of equal width/ length, $\Delta x = \frac{b-a}{n} = \frac{1}{4}$







• Total area of the 4 rectangles: $(n=4)$

max value of $f(x)$ in 2nd subinterval

$$1 \cdot \frac{1}{4} + \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} = \frac{25}{32}$$

$$= 0.78125$$

Δx

• The height of each rectangle is the maximum (upper most) value of $f(x) = 1 - x^2$ in each subinterval.

• So $\frac{25}{32}$ is an upper sum

• Total area of rectangles:

min value of $f(x)$ in 2nd subinterval

$$\frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} = \frac{17}{32}$$

$$= 0.53125$$

• Height of each rectangle is the minimum (lowermost) value of $f(x) = 1 - x^2$ in each subinterval.

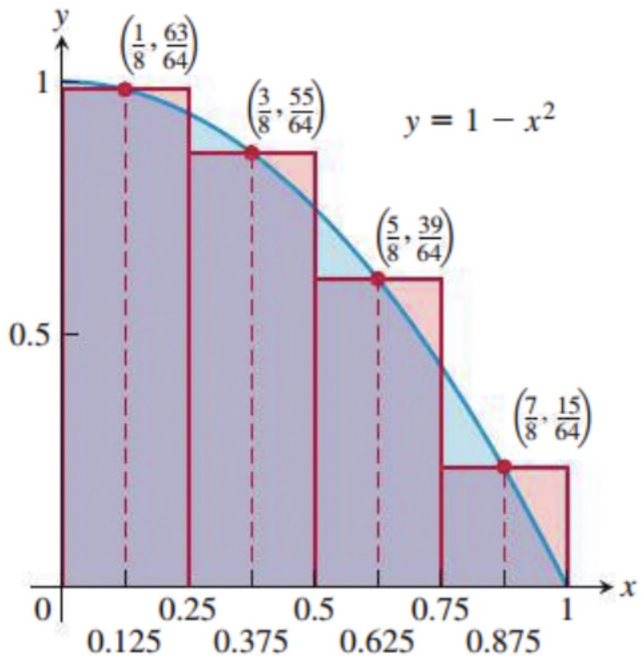
• So $\frac{17}{32}$ is a lower sum

(upper sum) (lower sum)

* $0.78125 >$ (Exact area of region R) > 0.53125

* The error cannot be bigger than (upper sum) − (lower sum) $= 0.25$

- The <u>midpoint rule</u> is using the value of $f(x)$ at the midpoint of each subinterval (for the height of the rectangles).



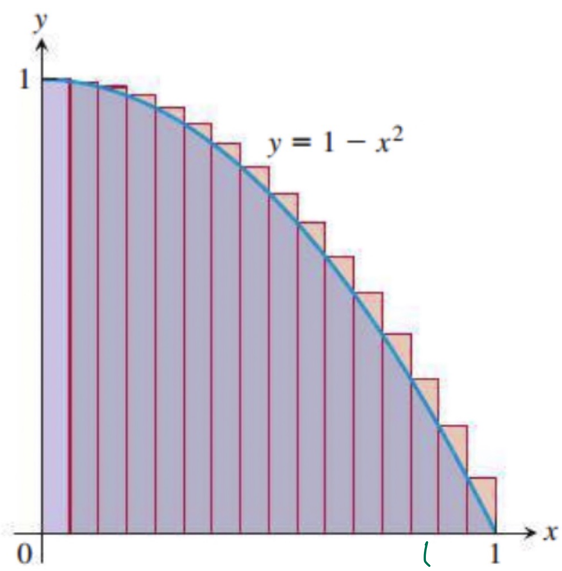
- Total area of $n=4$ rectangles :

$$\underbrace{\frac{63}{64}}\cdot\frac{1}{4} + \frac{55}{64}\cdot\frac{1}{4} + \frac{39}{64}\cdot\frac{1}{4} + \frac{15}{64}\cdot\frac{1}{4} = 0{,}671875$$

"midpoint" value of $f(x)$ in 2nd subinterval

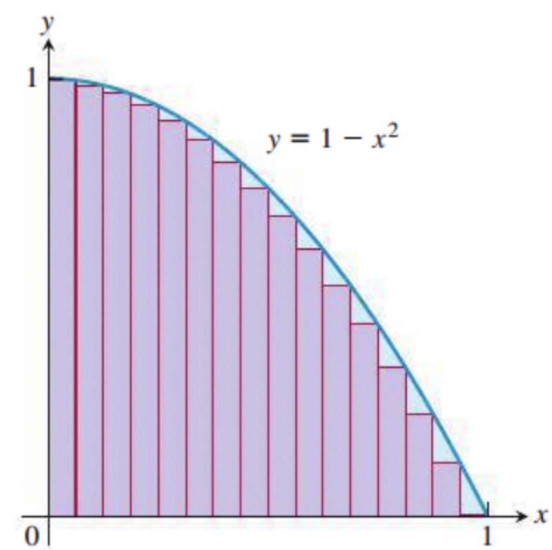
- With the midpoint rule, we cannot be sure if the approximation is bigger / smaller than the actual.

- As we take more rectangles (thinner subintervals), we get better & better approximations of the area of R.



$$\Delta x = \frac{b-a}{16} = \frac{1}{16}$$

upper sum using $n=16$
0,634765625



lower sum using $n=16$
0,697265625

- Taking greater & greater n, the sums seem to approach $\frac{2}{3}$.

② Distance Traveled

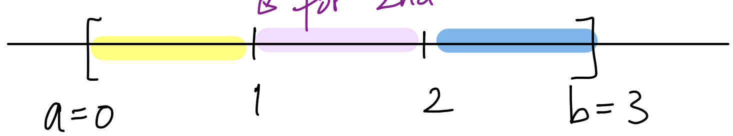
Ex: The velocity of an object fired straight into the air is $f(t) = 160 - 9.8t$ m/sec. Estimate the distance traveled by the object (how far it rises) during the first 3 sec, using $n=3$ subintervals and left-endpoint values.

Answer:

- Our time interval is $[a,b] = [0,3]$.
- Partition it into $n=3$ subintervals of equal width:

$$\Delta t = \frac{a-b}{n} = \frac{3-0}{3} = 1$$

left-endpoint for 2nd subinterval



- For each subinterval, use the distance formula

distance traveled = constant velocity × time

- Approximation of distance traveled is

$$f(0)\cdot\Delta x + f(1)\cdot\Delta x + f(2)\cdot\Delta x =$$

$$(160-0)1 + (160-9.8)1 + (160-9.8(2))1 =$$

$$\boxed{450.6 \text{ m}}$$

③ Average value of a nonnegative continuous function.

- Average value of a collection of numbers $x_1, x_2, \dots, x_n$

  is $\dfrac{x_1 + x_2 + \dots + x_n}{n}$.

- What is the average value of a continuous function f on an interval $[a,b]$?

  E.g. What does it mean to say "average temperature in Lowell today is 70 degrees"?

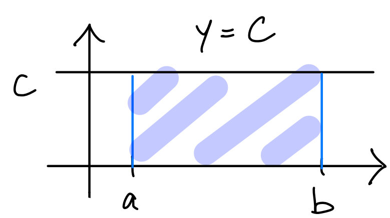
- If the function is constant on the interval $[a,b]$,

  $f(x) = c$ for x in $[a,b]$,

  (e.g. $f(x) = 5$ for x in $[2,6]$)

  then the average is c.

- Graph of $f(x)$ over $[a,b]$ is a rectangle:

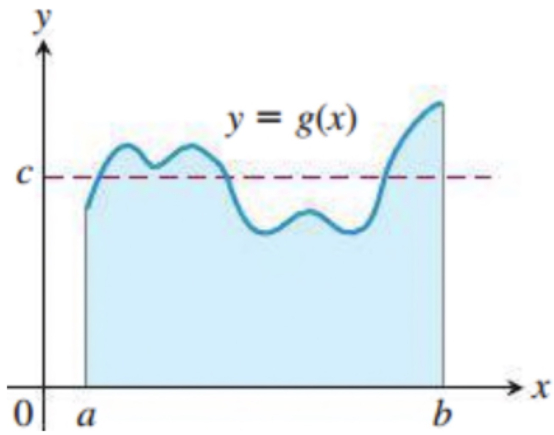


- We can interpret this average as

$$\frac{(\text{area of rectangle})}{(\text{width of rectangle})} = \frac{c(b-a)}{b-a} = c$$

- If the function is nonconstant

E.g.



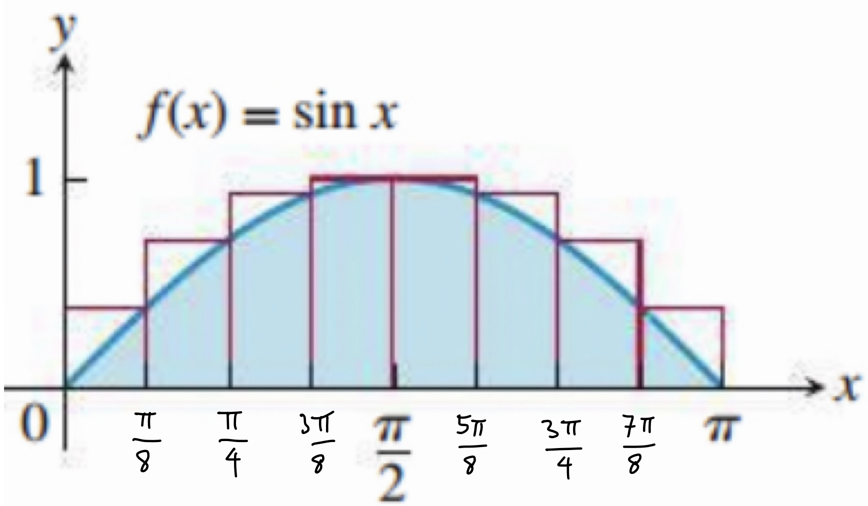
- Think of this graph as some water sloshing around in a tank between enclosing walls at $x=a$ & $x=b$.
- To get the average height of the water, let it settle down until its height is constant.
- Height $= \dfrac{\text{area under the graph of } g(x)}{b-a}$

- So define <u>AVERAGE VALUE</u> of a nonnegative function on an interval $[a,b]$ to be

$$\frac{\text{area under its graph}}{b-a}.$$

EX: Estimate the average value of $f(x) = \sin x$ on the interval $[0, \pi]$ by partitioning the interval into $n = 8$ subintervals of equal length and using an upper sum.

Answer:



Because $f(x)$ is increasing on $[0, \frac{\pi}{2}]$, we use the right endpoints to compute height of rectangle

Use left endpoints

- Width of each subinterval is $\Delta x = \dfrac{b-a}{n} = \dfrac{\pi - 0}{8} = \dfrac{\pi}{8}$
- Upper sum approximation of the area is:

$$\underbrace{\sin\left(\frac{\pi}{8}\right)\cdot\frac{\pi}{8} + \sin\left(\frac{\pi}{4}\right)\cdot\frac{\pi}{8} + \sin\left(\frac{3\pi}{8}\right)\cdot\frac{\pi}{8} + \sin\left(\frac{\pi}{2}\right)\cdot\frac{\pi}{8}}_{\text{total area of the first 4 rectangles}}$$

$$+ \underbrace{\sin\left(\frac{\pi}{2}\right)\cdot\frac{\pi}{8} + \sin\left(\frac{5\pi}{8}\right)\cdot\frac{\pi}{8} + \sin\left(\frac{3\pi}{4}\right)\cdot\frac{\pi}{8} + \sin\left(\frac{7\pi}{8}\right)\frac{\pi}{8}}_{\text{Last 4 rectangles}} \approx 2.364$$

- Estimated average value of $f(x) = \sin x$ on $[0, \pi]$ is $\boxed{\dfrac{2.364}{\pi - 0}}$
- This estimate is greater than actual average.
- As we increase the number of rectangles, the estimate gets close to $\dfrac{2}{\pi}$.