4.8 Antiderivatives

Idea: recover a function from its derivative

Given a function f, find a function F whose derivative is f.

DEFINITIONS A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Ex : Find an antiderivative for ...

a.) $f(x) = 2x$

Think backward. What function do we know has a derivative equal to $2x$?

How about x^2, $x^2 + 5$, $x^2 - 100$?

Check: The derivative of each of these is $2x$

b.) $g(x) = -\sin(x)$

$G(x) = \cos(x)$ is an antiderivative of $g(x) = -\sin(x)$.

So is $\cos(x) - 47$, $\cos(x) + 60$, etc

THEOREM 8 If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

This is a family of functions

They all have identical derivative, the function f.

* Review differentiation rules from Chapter 3

* Review Sec 3.3

- For any real number n except -1,

$$\frac{d}{dx}(x^n) = n x^{n-1} \quad \text{or} \quad \frac{d}{dx}\Big(\underbrace{\frac{1}{n}x^n}_{\text{an antiderivative}}\Big) = \frac{1}{n} n x^{n-1} = x^{n-1}.$$

our original function

Ex: $f(x) = x^5$

General antiderivative of $f(x)$ is $F(x) = \frac{1}{6}x^6 + C$

Ex: $g(x) = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

General antiderivative of $g(x)$ is $G(x) = \frac{1}{\left(\frac{1}{2}\right)} x^{-\frac{1}{2}+1} + C$

$$= 2x^{\frac{1}{2}} + C$$

$$= 2\sqrt{x} + C$$

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* Review Sec 3.3 & 3.8

- For any nonzero k, $\frac{d}{dx}(e^{kx}) = k e^{kx}$ or $\frac{d}{dx}\Big(\underbrace{\frac{1}{k}e^{kx}}_{\text{an antiderivative}}\Big) = \frac{1}{k} k e^{kx} = \underbrace{e^{kx}}_{\text{original function}}$

- For any positive a except for 1,

$$\frac{d}{dx}(a^x) = \frac{d}{dx}\left(e^{\ln(a^x)}\right)$$

$$= \frac{d}{dx}\left(e^{x \ln(a)}\right)$$

Here $\underbrace{\ln(a)}_{\text{nonzero because } a \neq 1}$ plays the role of k in e^{kx}

$$= \ln(a)\, e^{x \ln(a)}$$

$$= \ln(a)\, a^x$$

$$\text{or} \quad \frac{d}{dx}\Big(\underbrace{\frac{1}{\ln(a)} a^x}_{\text{an antiderivative}}\Big) = \frac{1}{\ln(a)} \ln(a)\, a^x = \underbrace{a^x}_{\text{original function}}$$

Ex: $f(x) = 2^x$

General antiderivative of $f(x)$ is $F(x) = \frac{1}{\ln(2)} 2^x + C$ Here $a = 2$

* Review Sec 3.5

For any nonzero k,

$$\frac{d}{dx}(\sin kx) = k\cos x$$

$$\frac{d}{dx}(\cos kx) = -\sin kx$$

$$\frac{d}{dx}(\tan(kx)) = (\sec kx)^2 k$$

$$\frac{d}{dx}(\cot(kx)) = -(\csc(kx))^2 k$$

$$\frac{d}{dx}(\sec(kx)) = \sec(kx)\tan(kx)\,k$$

$$\frac{d}{dx}(\csc(kx)) = -\csc(kx)\cot(kx)\,k$$

Ex: $f(x) = \cos\left(\frac{1}{2}x\right)$ Here $k=\frac{1}{2}$

General antiderivative of $f(x)$ is $F(x) = \dfrac{\sin\left(\frac{1}{2}x\right)}{\left(\frac{1}{2}\right)} + C = 2\sin\left(\frac{x}{2}\right)+C$

TABLE 4.2 **Antiderivative formulas, *k* a nonzero constant**

Function	General antiderivative	Function	General antiderivative
1. x^n	$\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$	**8.** e^{kx}	$\frac{1}{k}e^{kx} + C$
2. $\sin kx$	$-\frac{1}{k}\cos kx + C$	**9.** $\frac{1}{x}$	$\ln\lvert x\rvert + C, \quad x \neq 0$
3. $\cos kx$	$\frac{1}{k}\sin kx + C$	**10.** $\frac{1}{\sqrt{1-k^2x^2}}$	$\frac{1}{k}\sin^{-1}kx + C$
4. $\sec^2 kx$	$\frac{1}{k}\tan kx + C$	**11.** $\frac{1}{1+k^2x^2}$	$\frac{1}{k}\tan^{-1}kx + C$
5. $\csc^2 kx$	$-\frac{1}{k}\cot kx + C$	**12.** $\frac{1}{x\sqrt{k^2x^2-1}}$	$\sec^{-1}kx + C, kx > 1$
6. $\sec kx \tan kx$	$\frac{1}{k}\sec kx + C$	**13.** a^{kx}	$\left(\frac{1}{k\ln a}\right)a^{kx} + C, a > 0, a \neq 1$
7. $\csc kx \cot kx$	$-\frac{1}{k}\csc kx + C$		

- Recall: Derivatives are linear
- Antiderivatives are also <u>linear</u>:

FACT

If $F(x)$ is an antiderivative of $f(x)$
and $G(x)$ is an antiderivative of $g(x)$,

then

1. $kF(x) + C$ is the general antiderivative of $kf(x)$ for any number k ("constant multiple rule")

2. $F(x) + G(x) + C$ is the general antiderivative of $f(x) + g(x)$ ("sum rule")

Ex: $f(x) = \dfrac{3}{\sqrt{x}} + \cos\left(\frac{1}{2}x\right)$

Earlier we computed general antiderivatives of

$\dfrac{1}{\sqrt{x}}$ and $\cos\left(\frac{1}{2}x\right)$.

Using the fact that antiderivatives are linear, we get that the general antiderivative of $f(x)$ is

$$3\left(2\sqrt{x}\right) + 2\sin\left(\frac{x}{2}\right) + C$$

$$6\sqrt{x} + 2\sin\left(\frac{x}{2}\right) + C$$

Indefinite Integrals

A special symbol is used to denote the collection of all antiderivatives of a function f.

DEFINITION The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x; it is denoted by

$$\int \underbrace{f(x)}_{\text{integrand}}\, dx.$$

x is the variable of integration

The symbol $\int$ is an $\boxed{\textbf{integral sign}}$. The function f is the **integrand** of the integral, and x is the **variable of integration**.

Ex: Evaluate $\displaystyle\int \left(\frac{1}{x} + 2e^{2x}\right) dx$

Answer: • This is the same as asking "Given a function $f(x) = \frac{1}{x} + 2e^{2x}$, find the most general antiderivative of $f(x)$."

• It's enough to find one antiderivative and add an arbitrary constant C.

Review Sec 3.8

• Check: $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$, $\frac{d}{dx}(e^{2x}) = 2e^{2x}$

• So $\ln|x| + e^{2x}$ is an antiderivative of $f(x)$.

• So

$$\int\left(\frac{1}{x} + 2e^{2x}\right) dx = \overbrace{\ln|x| + e^{2x}}^{\text{an antiderivative}} + \underbrace{C}_{\text{arbitrary constant}}$$