10.4 Graphing polar coordinate equations

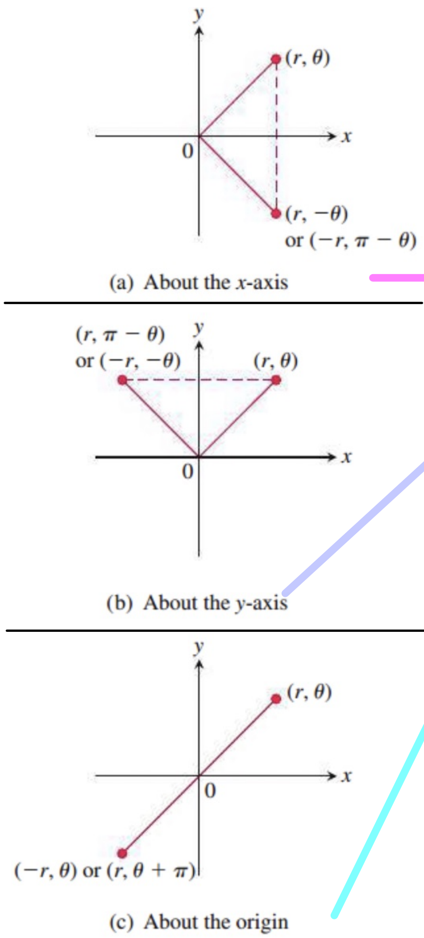

FIGURE 10.28 Three tests for symmetry in polar coordinates.

Symmetry Tests for Polar Graphs in the Cartesian xy-Plane

1. *Symmetry about the x-axis:* If the point (r, θ) lies on the graph, then the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph (Figure 10.28a).
2. *Symmetry about the y-axis:* If the point (r, θ) lies on the graph, then the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph (Figure 10.28b).
3. *Symmetry about the origin:* If the point (r, θ) lies on the graph, then the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph (Figure 10.28c).

The slope of a polar curve in the xy-plane

The slope of a polar curve $r = f(\theta)$ in the xy-plane is $\frac{dy}{dx}$. This is <u>not</u> $\frac{df}{d\theta}$.

We can think of this curve as the graph of "parametric equations":

$x(\theta) = r\cos\theta = f(\theta)\cos\theta$ and $y(\theta) = r\sin\theta = f(\theta)\sin\theta$

from Sec 10.3 — because $r = f(\theta)$ — from Sec 10.3 — because $r = f(\theta)$

Chain Rule

$$\frac{d}{d\theta} y(x(\theta)) = \frac{dy}{dx}\frac{dx(\theta)}{d\theta} \quad \text{or} \quad \frac{dy}{d\theta} = \frac{dy}{dx}\frac{dx}{d\theta} \text{ for short}$$

Then $\boxed{\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}}$

or

$$\boxed{\frac{dy}{dx} = \frac{\frac{d}{d\theta}\left(f(\theta)\sin\theta\right)}{\frac{d}{d\theta}\left(f(\theta)\cos\theta\right)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}}$$

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Chain Rule

$$\frac{d}{d\theta} y(x(\theta)) \overset{\text{Chain Rule}}{=} \frac{dy}{dx}\,\frac{dx(\theta)}{d\theta} \quad \text{or} \quad \frac{dy}{d\theta} = \frac{dy}{dx}\,\frac{dx}{d\theta} \text{ for short}$$

Then

$$\boxed{\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}}$$

Ex:

Find the slope of the tangent line of the curve $r = 1 + \sin\theta$ at the point $\theta = \frac{\pi}{3}$

Solution

Sec 10.3

$$x = r\cos\theta = (1 + \sin\theta)\cos\theta = \cos\theta + \tfrac{1}{2}\sin 2\theta$$

$$y = r\sin\theta = (1 + \sin\theta)\sin\theta = \sin\theta + \sin^2\theta$$

Then we have

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta + 2\sin\theta\,\cos\theta}{-\sin\theta + \cos 2\theta} = \frac{\cos\theta + \sin 2\theta}{-\sin\theta + \cos 2\theta}$$

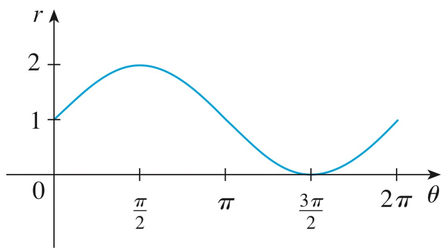
(a) The slope of the tangent at the point where $\theta = \pi/3$ is

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \frac{\cos\left(\frac{\pi}{3}\right) + \sin\left(\frac{2\pi}{3}\right)}{-\sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{2\pi}{3}\right)} = \frac{\frac{1}{2} + \frac{\sqrt{3}}{2}}{-\frac{\sqrt{3}}{2} - \frac{1}{2}} = \boxed{-1}$$

Graphing in Polar Coordinates

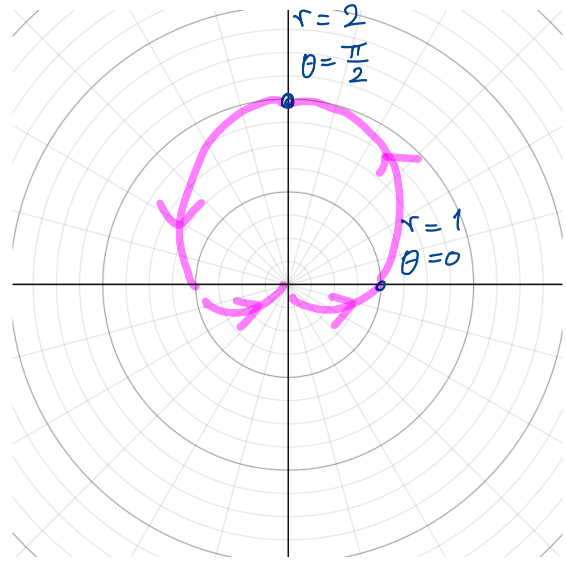
Example Sketch the polar equation $r = 1 + \sin\theta$.

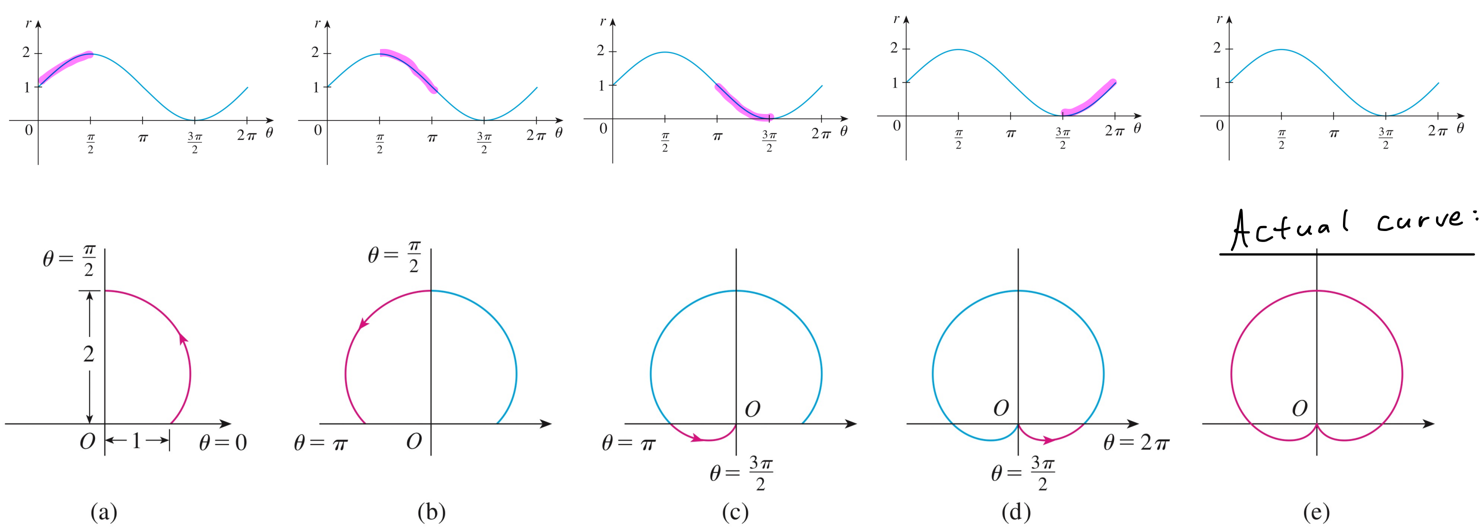
Sol:



$r = 1 + \sin\theta$ in Cartesian coordinates, $0 \leqslant \theta \leqslant 2\pi$

Actual curve:



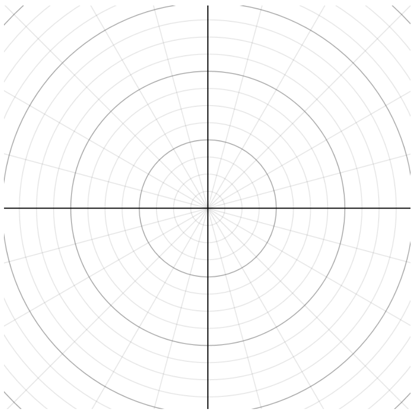


Stages in sketching the cardioid $r = 1 + \sin\theta$

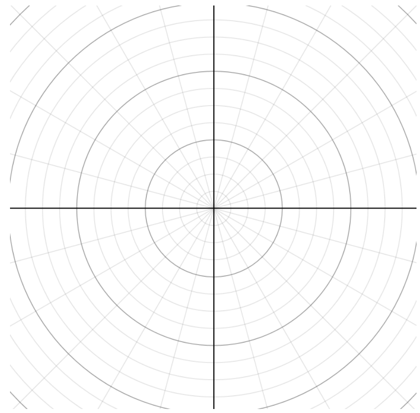
Note: This curve is symmetric about the y-axis

The graph of a polar equation $r = f(\theta)$ consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

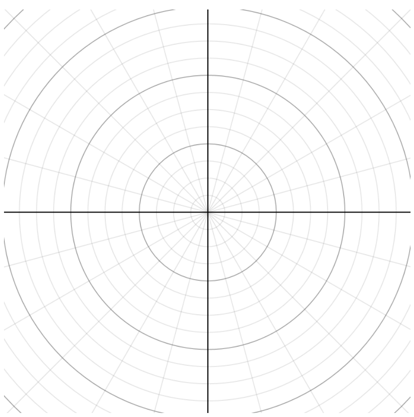
Exercise Sketch the following polar equations.



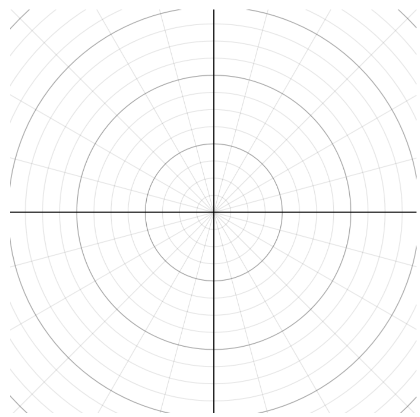
(a) $r = 4$



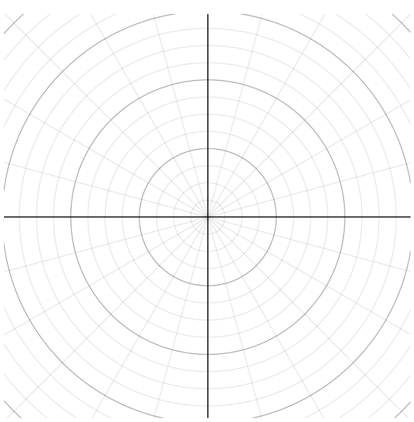
(b) $\theta = \dfrac{\pi}{6}$



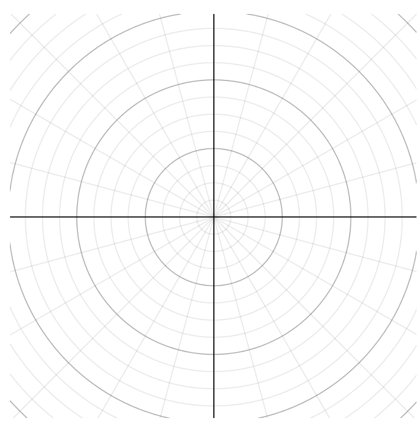
(c) $r = 4\cos\theta$



(d) $r = 4\sin\theta$



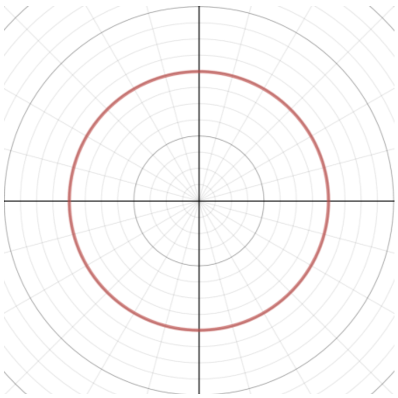
(e) $r = 2 + 2\sin\theta$



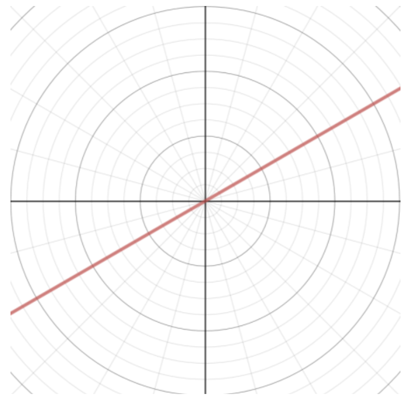
(f) $r = 4\cos 2\theta$

The graph of a polar equation $r = f(\theta)$ consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

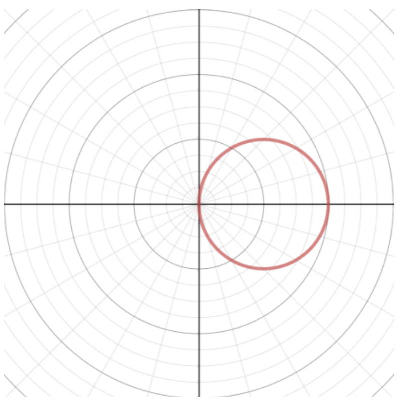
Exercise (Answer Key) Sketch the following polar equations.



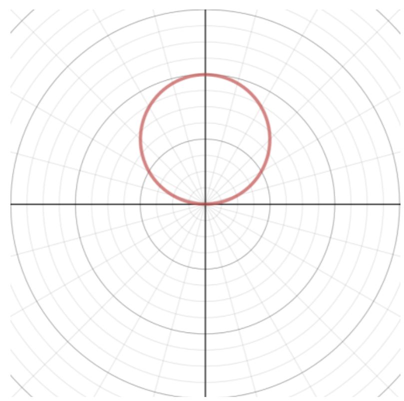
(a) $r = 4$



(b) $\theta = \dfrac{\pi}{6}$



(c) $r = 4\cos\theta$



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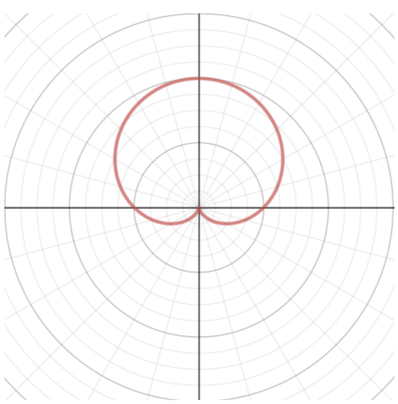
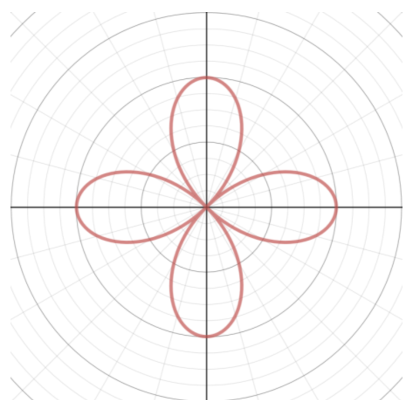


(e) $r = 2 + 2\sin\theta$

Cardioid



(f) $r = 4\cos 2\theta$

Four-leaved Rose

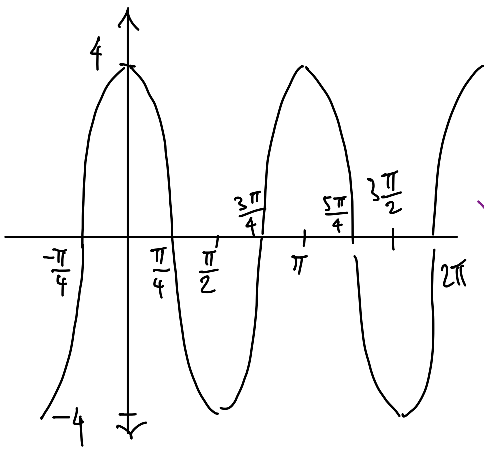
Example

Sketch the curve $r = 4\cos(2\theta)$

Sol:

In Cartesian coordinates

$r(\theta) = 4\cos(2\theta)$



Actual curve: