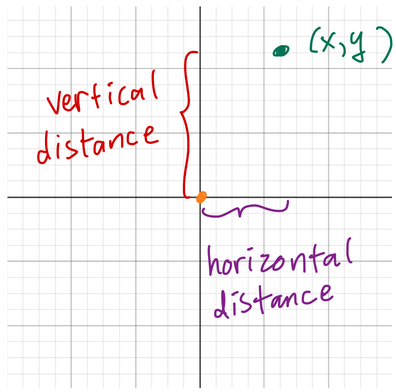
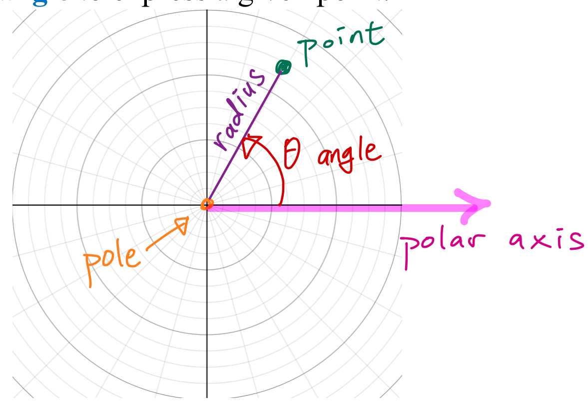
Lecture 10.3 **Polar Coordinates**

Polar Coordinates

(In Multivariable Calculus: polar coordinates → cylindrical and spherical coordinates in 3D)

Instead of using **horizontal distance** and **vertical distance** from the axes, we use the distance from the origin (**radius**) as well as the corresponding **angle** to express a given point.





- The origin is called the **pole**.
- The positive x-axis is called the **polar axis**.
- The polar coordinates for a point P have the form (r,θ), where …
  - r is the distance from the origin to P, and
  - θ is an angle starting from the $\underbrace{\text{positive } x\text{-axis}}_{\text{polar axis}}$ and ending on the ray that passes through the pole and the point P.

- Positive angles are measured **counterclockwise** from the positive x-axis.

Example

Caution

The representation for a given point is **not unique** in the polar coordinate system.

(r,θ), $(r, \theta + \underbrace{2n\pi}_{})$ and $(-r, \theta + \underbrace{(2n+1)\pi}_{})$ refer to the same point.

any even multiple of π

changing angle by $180°$

Example:

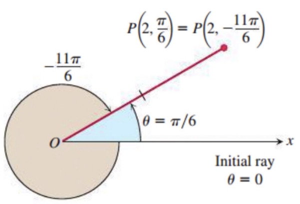


FIGURE 10.21 Polar coordinates are not unique.

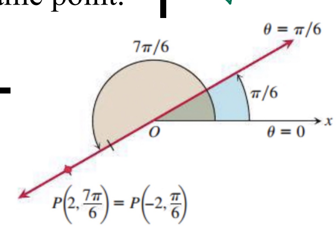
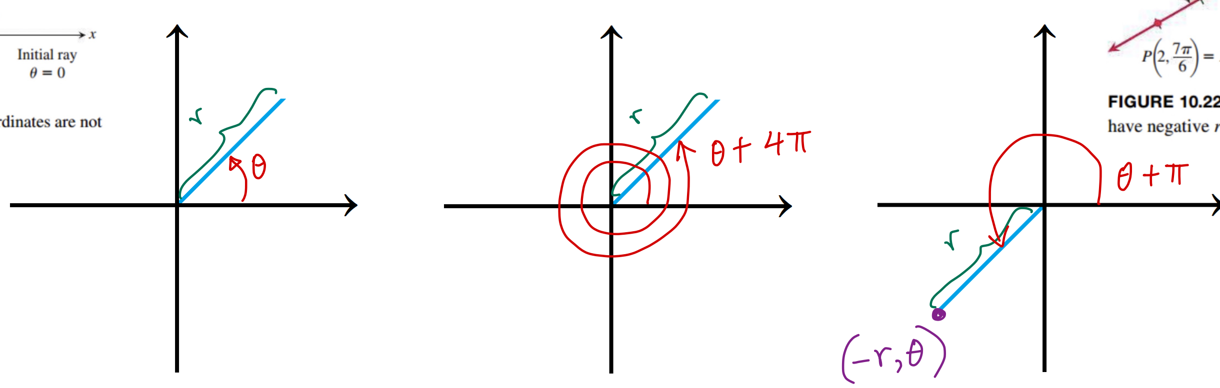


FIGURE 10.22 Polar coordinates can have negative r-values.



The origin is specified as $(0,\theta)$ in polar coordinates, where θ is any angle.

$\cos(\pi+\theta) = -\cos(\theta)$

$\sin(\pi+\theta) = -\sin(\theta)$

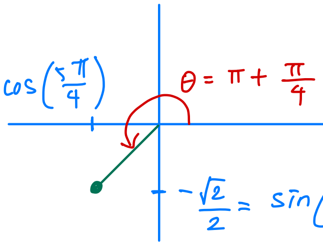
Lecture 10.3 **Polar Coordinates**

$5\frac{\pi}{4} = \pi + \frac{\pi}{4}$

Find other representations for the same point

Graph the point $\left(1, \frac{5\pi}{4}\right)$ in polar coordinates. Give ~~two~~ four alternative representations for the point.

$-\frac{\sqrt{2}}{2} = \cos\left(\frac{5\pi}{4}\right)$ $\quad \theta = \pi + \frac{\pi}{4}$



$-\frac{\sqrt{2}}{2} = \sin\left(\frac{5\pi}{4}\right)$

$\left(1, \frac{5\pi}{4} + 2\pi\right), \left(1, \frac{5\pi}{4} - 2\pi\right)$

$\left(-1, \frac{5\pi}{4} + \pi\right), \left(-1, \frac{5\pi}{4} - 3\pi\right)$

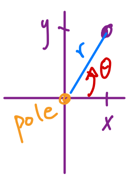
Converting Between Cartesian and Polar Coordinates

Procedure **Converting Coordinates**

A point with polar coordinates (r, θ) has Cartesian coordinates (x, y), where

$x = \left(\begin{array}{c}\text{horizontal}\\ \text{distance}\end{array}\right) = r\cos\theta \quad , \quad y = \left(\begin{array}{c}\text{vertical}\\ \text{distance}\end{array}\right) = r\sin\theta$

A point with Cartesian coordinates (x, y) has polar coordinates (r, θ), where



$r = \left(\begin{array}{c}\text{distance}\\ \text{from pole}\end{array}\right) = \sqrt{x^2 + y^2} \quad , \quad \tan(\theta) = \frac{y}{x}$

or $r^2 = x^2 + y^2$

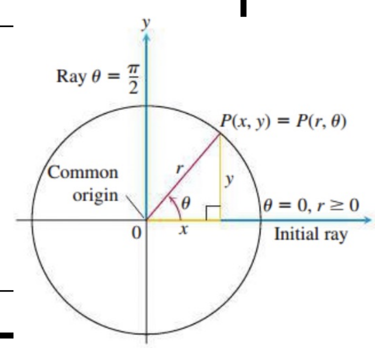


FIGURE 10.26 The usual way to relate polar and Cartesian coordinates.

Polar to Cartesian

Express the point with polar coordinates $\left(\overset{r}{2}, \overset{\theta}{\frac{3\pi}{4}}\right)$ in Cartesian coordinates.

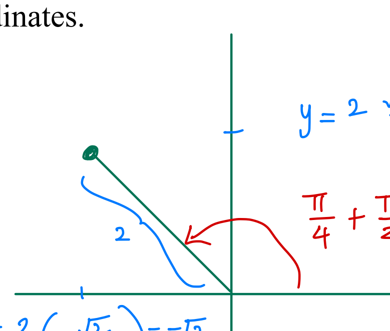
$x = r\cos\theta = 2\cos\frac{3\pi}{4} = -\sqrt{2}$

$y = r\sin\theta = 2\sin\frac{3\pi}{4} = \sqrt{2}$

Answer: $(-\sqrt{2}, \sqrt{2})$

$y = 2\frac{\sqrt{2}}{2} = \sqrt{2}$

$\frac{\pi}{4} + \frac{\pi}{2}$



$x = 2\left(-\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$

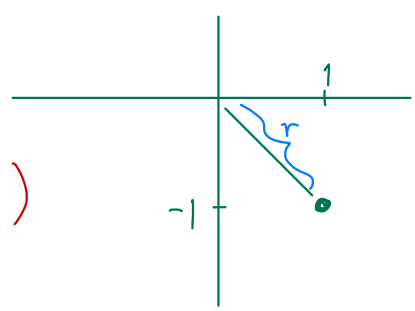
Cartesian to polar

Express the point with Cartesian coordinates $(1, -1)$ in polar coordinates.

$r = \sqrt{x^2 + y^2} = \sqrt{1+1} = \sqrt{2}$

$\theta = -\frac{\pi}{4}$ or $-\frac{\pi}{4} + 2n\pi$

(from picture or compute $\theta = \arctan\left(\frac{-1}{1}\right) = -\frac{\pi}{4}$)

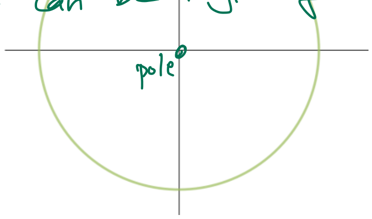


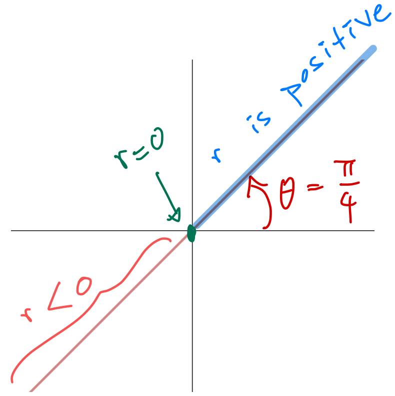
Converting Between Cartesian and Polar Equations

A curve in polar coordinates is the set of points that satisfy an equation in r and θ. Some sets of points are easier to describe in polar coordinates than in Cartesian coordinates.

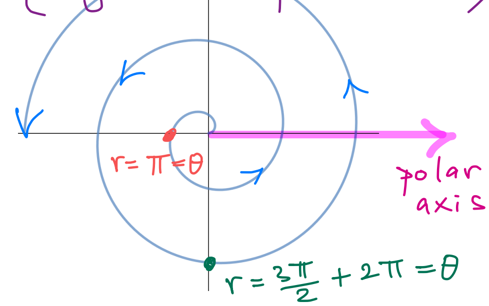
Examples:

The set of all points with distance 3 from the pole, angle θ can be anything





All points where (distance from pole) = (angle from polar axis)



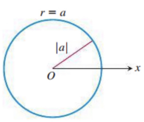


FIGURE 10.24 The polar equation for a circle is $r = a$.

polar equation $r = 3$

Also $r = -3$

polar equation $\theta = \frac{\pi}{4}$

Also $\theta = \frac{5\pi}{4}$

polar equation $r = \theta$

A circle or a line can have more than one polar equation!

(Graph with Desmos: https://www.desmos.com/calculator/j6ha36k9zi)

Polar to Cartesian

Convert the polar equation $r = 8\sin\theta$ to a Cartesian equation.

Page 2: $x = r\cos\theta$, $y = r\sin\theta \iff \frac{y}{r} = \sin\theta$

Plug in $\frac{y}{r} = \sin\theta$ into $r = 8\sin\theta$: $r = 8\frac{y}{r} \iff r^2 = 8y$

Page 2: $r^2 = x^2 + y^2$

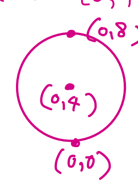
Plug in $r^2 = x^2 + y^2$ into $r^2 = 8y$: $\boxed{x^2 + y^2 = 8y}$

$x^2 + y^2 - 8y = 0$

$x^2 + y^2 - 8y + 4^2 = 4^2$

$x^2 + (y-4)^2 = 4^2$

circle with radius 4, centered at (0,4)

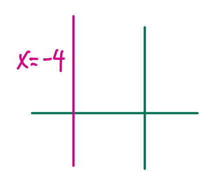


Polar to Cartesian

Convert the polar equation $r\cos\theta = -4$ to a Cartesian equation.

Page 2: $x = r\cos\theta$

Plug in $x = r\cos\theta$ into $r\cos\theta = -4$: $\boxed{x = -4}$



Cartesian to polar equation

Convert the Cartesian equation $x^2(x^2 + y^2) = 4y^2$ to a polar equation.

Page 2: $r^2 = x^2 + y^2$

Plug into $x^2(x^2 + y^2) = 4y^2$: $x^2 r^2 = 4y^2$

Page 2: $x = r\cos\theta$, $y = r\sin\theta$

Plug into $x^2 r^2 = 4y^2$: $r^2(\cos\theta)^2 r^2 = 4r^2(\sin\theta)^2$

$r^2 = 4(\tan\theta)^2$

$\boxed{r = 2\tan\theta}$

Additional examples going between polar & cartesian

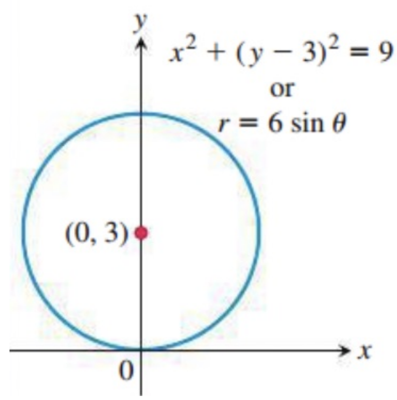
EXAMPLE 4 Here are some plane curves expressed in terms of both polar coordinate and Cartesian coordinate equations.

Polar equation	Cartesian equivalent
$r \cos \theta = 2$	$x = 2$
$r^2 \cos \theta \sin \theta = 4$	$xy = 4$
$r^2 \cos^2\theta - r^2 \sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r \cos \theta$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos \theta$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

Some curves are more simply expressed with polar coordinates; others are not.

Example: Find a polar equation for the circle

$$x^2 + (y-3)^2 = 9$$



Sol:

$$x^2 + (y-3)^2 = 9$$

$$x^2 + y^2 - 6y + 9 = 9 \qquad \text{Expand } (y-3)^2.$$

$$x^2 + y^2 - 6y = 0 \qquad \text{Cancelation}$$

$$r^2 - 6r \sin \theta = 0 \qquad x^2 + y^2 = r^2,\ y = r \sin \theta$$

$$r = 0 \quad \text{or} \quad r - 6 \sin \theta = 0$$

$$r = 6 \sin \theta \qquad \text{Includes both possibilities}$$

Additional examples going between polar & cartesian

EXAMPLE 6 Replace the following polar equations by equivalent Cartesian equations and identify their graphs.

(a) $r\cos\theta = -4$

(b) $r^2 = 4r\cos\theta$

(c) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

Solution We use the substitutions $r\cos\theta = x$, $r\sin\theta = y$, and $r^2 = x^2 + y^2$.

(a) $r\cos\theta = -4$

The Cartesian equation: $r\cos\theta = -4$

$x = -4$ Substitute.

The graph: Vertical line through $x = -4$ on the x-axis

(b) $r^2 = 4r\cos\theta$

The Cartesian equation: $r^2 = 4r\cos\theta$

$x^2 + y^2 = 4x$ Substitute.

$x^2 - 4x + y^2 = 0$

$x^2 - 4x + 4 + y^2 = 4$ Complete the square.

$(x-2)^2 + y^2 = 4$ Factor.

The graph: Circle, radius 2, center $(h, k) = (2, 0)$

(c) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

The Cartesian equation: $r(2\cos\theta - \sin\theta) = 4$

$2r\cos\theta - r\sin\theta = 4$ Multiply by r.

$2x - y = 4$ Substitute.

$y = 2x - 4$ Solve for y.

The graph: Line, slope $m = 2$, y-intercept $b = -4$ ∎

Defining regions, segments, rays

EXAMPLE 3 Graph the sets of points whose polar coordinates satisfy the following conditions.

(a) $1 \le r \le 2$ and $0 \le \theta \le \dfrac{\pi}{2}$

(b) $-3 \le r \le 2$ and $\theta = \dfrac{\pi}{4}$

(c) $\dfrac{2\pi}{3} \le \theta \le \dfrac{5\pi}{6}$ (no restriction on r)

Sol: