7.8 Improper Integrals (Type 1)

An application

(Source: Textbook Sec 5.4 Exercise 31)

Work required to move a rocket from the surface of earth to another point t meters (from the center of earth) is

$$\int_R^t G\,m\,M \frac{1}{x^2}\,dx$$ in joule or newton-meter,

where G is a constant in $\frac{\text{newton meters}^2}{\text{kg}^2}$ (called gravitational constant),

m is the mass in kg of the rocket,

M is the mass in kg of earth,

R is the radius of earth.

Evaluate:

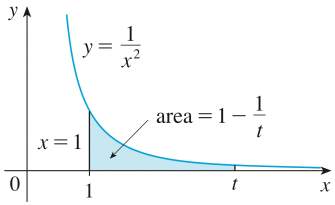
$$\boxed{GmM\int_R^t x^{-2}\,dx = GmM\left.\frac{x^{-1}}{-1}\right|_R^t = GmM\left(-\frac{1}{t}+\frac{1}{R}\right)}$$

The work required to free the rocket from earth's gravity pull

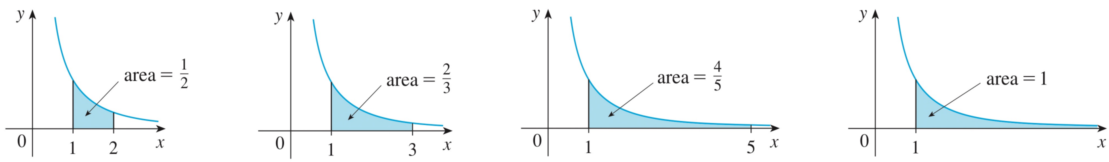
is $$\underbrace{\lim_{t\to\infty}\int_R^t \frac{GmM}{x^2}\,dx}_{} = \lim_{t\to\infty} GmM\left(-\frac{1}{t}+\frac{1}{R}\right) = GmM\left(\frac{1}{R}\right)$$

Vocab: this concept is called an improper integral, even when the limit does not exist

Ex 1



$$A(t) = \int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_1^t = 1 - \frac{1}{t}$$



$\int_1^2 \frac{1}{x^2}dx = 1 - \frac{1}{2}$ $\int_1^3 \frac{1}{x^2}dx = 1 - \frac{1}{3}$ $\int_1^4 \frac{1}{x^2}dx = 1 - \frac{1}{4}$ $\lim_{t\to\infty}\int_1^t \frac{1}{x^2}dx = \lim_{t\to\infty} 1 - \frac{1}{t}$

$= 1 - 0$

New notation: $\int_1^\infty \frac{1}{x^2}dx \overset{\text{Def}}{=} \lim_{t\to\infty}\int_1^t \frac{1}{x^2}dx$ (if the right hand side exists)

---

1 Definition of an Improper Integral of Type 1

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geqslant a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leqslant b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

Example: $\int_1^\infty \frac{1}{x^2}dx \overset{\text{def}}{=} \lim_{t\to\infty}\int_1^t \frac{1}{x^2}dx$

since the RHS is equal to a number

- We say $\int_1^\infty \frac{1}{x^2}dx$ is $\boxed{\text{CONVERGENT}}$ because $\boxed{\lim_{t\to\infty}\int_1^t \frac{1}{x^2}dx \text{ equals a number.}}$

Ex 2

- $\int_t^{-1} \frac{1}{x}dx = \ln|x|\Big|_{x=t}^{x=-1} = \ln|-1| - \ln|t| = -\ln|t|$

So $\lim_{t\to-\infty}\int_t^{-1} \frac{1}{x}dx = \lim_{t\to-\infty} -\ln|t| = -\infty$ so $\int_{-\infty}^{-1} \frac{1}{x}dx$ is DIVERGENT

1 Definition of an Improper Integral of Type 1

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^{\infty} f(x)\,dx \overset{\text{def}}{=} \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$$

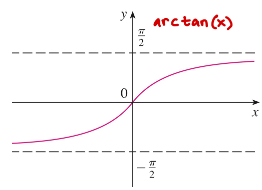
any number a will result in the same number

Warning • If $\int_a^\infty f(x)\,dx$ or $\int_{-\infty}^a f(x)\,dx$ is divergent, then we say $\int_{-\infty}^\infty f(x)\,dx$ is divergent.

• Both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ must be convergent for $\int_{-\infty}^\infty f(x)$ to be convergent.

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$ **Ex 3**

$$\int_0^t \frac{1}{1+x^2}\,dx = \arctan(x)\Big|_0^t = \arctan(t) - \underbrace{\arctan(0)}_{0}$$

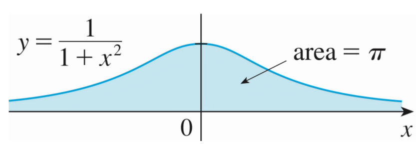


$$\lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \arctan(t) = \frac{\pi}{2}$$

$$\int_t^0 \frac{1}{1+x^2}\,dx = \arctan(x)\Big|_t^0 = \underbrace{\arctan(0)}_{0} - \arctan(t)$$

$$\lim_{t\to-\infty} \int_t^0 \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} -\arctan(t) = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

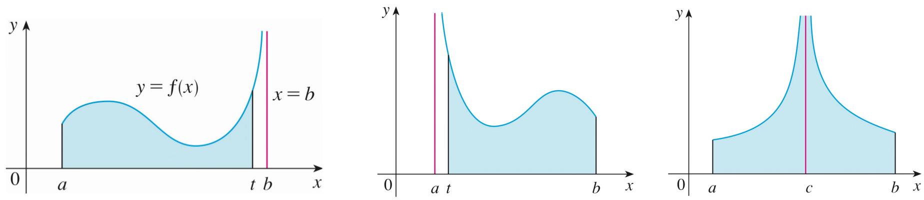
So $$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_0^\infty \frac{1}{1+x^2}\,dx + \int_{-\infty}^0 \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2}$$



7.8 Improper Integrals (Type 2)

The symbol $\int_a^b f(x)\,dx$ means an improper integral

if $f(x)$ has an infinite discontinuity in $[a,b]$



3 Definition of an Improper Integral of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

From now on, when you see $\int_a^b f(x)\,dx$, you need to first determine whether it's improper integral by checking for infinite discontinuity of f in $[a,b]$.

Find $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$ **Ex 4**

Because $\displaystyle\lim_{x\to 2^+} \frac{1}{\sqrt{x-2}} = \infty$, the function $\dfrac{1}{\sqrt{x-2}}$ has an infinite discontinuity in $[2,5]$, so the symbol $\int_2^5 \frac{1}{\sqrt{x-2}}dx$ means improper integral.

$$\int_t^5 \frac{1}{\sqrt{x-2}}dx = \int_t^5 (x-2)^{-\frac{1}{2}}dx = \frac{(x-2)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}\Bigg|_t^5 = 2\left(\sqrt{5-2} - \sqrt{t-2}\right)$$

$$\lim_{t\to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}} = \lim_{t\to 2^+} 2\left(\sqrt{3} - \sqrt{t-2}\right) = 2\sqrt{3}$$

So $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}} = 2\sqrt{3}$, so it is CONVERGENT.

Example

$\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}dx$ is an improper integral (new concept!)

$\displaystyle\int_4^7 \frac{1}{\sqrt{x-2}}dx$ is a usual integral (Calc 1 concept)

3 Definition of an Improper Integral of Type 2

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

If one of $\int_a^c f(x)\,dx$ or $\int_c^b f(x)\,dx$ is divergent, then $\int_a^b f(x)\,dx$ is divergent.

Evaluate $\displaystyle\int_0^3 \frac{dx}{x-1}$ if possible **Ex 5**

WARNING WRONG COMPUTATION WARNING

$$\int_0^3 \frac{1}{x-1}\,dx = \ln|x-1|\Big|_0^3 = \ln(2) - \underbrace{\ln(1)}_{0} = \ln(2)$$

Improper integrals are not integrals

Because $\displaystyle\lim_{x\to 1^+} \frac{1}{x-1} = \boxed{\infty}$ and $\displaystyle\lim_{x\to 1^-} \frac{1}{x-1} = \boxed{-\infty}$

the function $\dfrac{1}{x-1}$ has an infinite discontinuity at $x=1$,

so the symbol $\displaystyle\int_0^3 \frac{1}{x-1}dx$ means improper integral. (new concept! Sec 7.8)

Example: $\displaystyle\int_4^6 \frac{1}{x-1}dx$ is a usual integral (Calc 1 concept)

Need to check the convergence / divergence of

$\displaystyle\int_1^3 \frac{1}{x-1}dx$ and $\displaystyle\int_0^1 \frac{1}{x-1}dx$ separately.

$$\int_0^t \frac{1}{x-1}dx = \ln|x-1|\Big|_0^t = \ln|t-1| - \ln|0-1| = \ln|t-1| - \underbrace{\ln(1)}_{0}$$

$$\lim_{t\to 1^-} \int_0^t \frac{1}{x-1}dx = \lim_{t\to 1^-} \ln|t-1| = -\infty$$

So $\displaystyle\int_0^1 \frac{1}{x-1}dx = -\infty$, and we call $\displaystyle\int_0^1 \frac{1}{x-1}dx$ divergent.

Without even checking $\displaystyle\int_1^3 \frac{1}{x-1}dx$, we can say $\displaystyle\int_0^3 \frac{1}{x-1}dx$ is divergent.

Comparison Thm for Improper Integrals

IF $0 \le f(x) \le g(x)$ for $x \ge a$ and $\int_a^\infty \underbrace{g(x)}_{\text{"bigger"}}\,dx$ is convergent,

THEN $\int_a^\infty \underbrace{f(x)}_{\text{"smaller"}}\,dx$ is convergent

IF $0 \le f(x) \le g(x)$ for $x \ge a$ and $\int_a^\infty \underbrace{f(x)}_{\text{"smaller"}}\,dx = \infty$,

THEN $\int_a^\infty \underbrace{g(x)}_{\text{"bigger"}}\,dx = \infty$

(Vocab: We say $\int_a^\infty f(x)\,dx$ is divergent)

vocab: We say $\int_a^\infty g(x)\,dx$ is divergent

Determine whether $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ is convergent. **Ex 6**

(Note: $\int \frac{1+e^{-x}}{x}\,dx$ cannot be expressed as an elementary function, so we cannot evaluate it until we get to Sec 11.9–11.10)

Answer
- $0 \le \frac{1}{x} \le \frac{1+e^{-x}}{x}$ for $x \ge 1$
- $\int_1^\infty \frac{1}{x}\,dx \overset{\text{Def}}{=} \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln|x|\Big|_1^t = \lim_{t\to\infty} \ln|t| - \ln(1) = \infty$
- By the Comparison Thm for Improper Integrals,

$$\int_1^\infty \frac{1+e^{-x}}{x}\,dx = \infty$$

so it is not convergent.

(Another word for "not convergent" is divergent.)