

## SECTION 7.4 Integration of Rational Functions by Partial Fractions

### Outline

- a. Long division for polynomials
- b. Distinct linear factors: Webwork #1
- c. Repeated linear factors: Webwork #5
- d. Rationalizing substitution

# Def: Rational functions

Def A rational function is a ratio of polynomials

Example  $\frac{9x+3}{x^2-9}$  and  $\frac{6x^3+3x^2-45x-24}{x^2-9}$  are rational functions

Def A rational function  $\frac{P(x)}{Q(x)}$  is proper if  $\deg(P) < \deg(Q)$ ,  
otherwise it is improper.

Example  $\frac{9x+3}{x^2-9}$  is a proper rational function

$$\begin{aligned}\deg(P) &= 1 \\ \deg(Q) &= 2\end{aligned}$$

$\frac{6x^3+3x^2-45x-24}{x^2-9}$  is an improper rational function

$$\begin{aligned}\deg(P) &= 3 \\ \deg(Q) &= 2\end{aligned}$$

# Long division

- Goal of Sec 7.4: Integrate *any* proper rational function
- Any improper rational function  $\frac{P(x)}{Q(x)}$  can be written as

$$\left( \text{a polynomial } S(x) \right) + \frac{R(x)}{Q(x)}$$

can integrate using power rule

a proper rational function

Example: •  $\frac{6x^3 + 3x^2 - 45x - 24}{x^2 - 9}$

is an improper rational function.

- Perform long division

to write it as

$$S(x) + \frac{R(x)}{Q(x)}$$

← deg of  $R(x)$  smaller than deg of  $Q(x)$

a polynomial

$S(x)$  will go here

$$\underbrace{x^2 - 9}_{Q(x)} \overline{) \underbrace{6x^3 + 3x^2 - 45x - 24}_{P(x)}}$$

⋮

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$R(x)$  will go here

Long division

$$\frac{P(x)}{Q(x)} = \frac{6x^3 + 3x^2 - 45x - 24}{x^2 - 9}$$

$$\text{Goal: } \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

$$\textcircled{1} \frac{6x^3}{x^2} = 6x$$

$$6x$$

$$x^2 - 9$$

$$6x^3 + 3x^2 - 45x - 24$$

$$\textcircled{2} 6x(x^2 - 9) = 6x^3 - 54x \rightarrow$$

$$6x^3 \quad - 54x$$

Long division

$$\frac{P(x)}{Q(x)} = \frac{6x^3 + 3x^2 - 45x - 24}{x^2 - 9}$$

$$\text{Goal: } \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

$$\textcircled{1} \frac{6x^3}{x^2} = 6x \quad \downarrow$$

6x

$$\begin{array}{r} x^2 - 9 \overline{) 6x^3 + 3x^2 - 45x - 24} \\ \underline{6x^3} \phantom{+ 3x^2} - 54x \phantom{- 24} \\ \phantom{6x^3} + 9x - 24 \end{array}$$

$$\textcircled{2} 6x(x^2 - 9) = 6x^3 - 54x \rightarrow$$

$$\textcircled{3} \text{ Subtract with } \begin{array}{r} 6x^3 + 3x^2 - 45x - 24 \\ \underline{6x^3 \phantom{+ 3x^2} - 54x} \\ 3x^2 + 9x - 24 \end{array} \rightarrow$$

← deg is not smaller than deg of  $Q(x) = x^2 - 9$ , so keep going

Long division

$$\frac{P(x)}{Q(x)} = \frac{6x^3 + 3x^2 - 45x - 24}{x^2 - 9}$$

$$\text{Goal: } \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

$$\textcircled{1} \quad \frac{6x^3}{x^2} = 6x \quad \textcircled{1'} \quad \frac{3x^2}{x^2} = 3$$

$$6x + \boxed{3}$$

$$\begin{array}{r} x^2 - 9 \overline{) 6x^3 + 3x^2 - 45x - 24} \\ \underline{6x^3} \phantom{+ 3x^2} - 54x \phantom{- 24} \\ \phantom{6x^3} + 9x - 24 \end{array}$$

$$\textcircled{2} \quad 6x(x^2 - 9) = 6x^3 - 54x \rightarrow$$

$$\textcircled{3} \quad \begin{array}{l} \text{Subtract} \\ \text{with} \end{array} \quad \begin{array}{l} 6x^3 + 3x^2 - 45x - 24 \\ 6x^3 \phantom{+ 3x^2} - 54x \phantom{- 24} \end{array} \rightarrow 3x^2 + 9x - 24$$

$$\textcircled{2'} \quad \boxed{3}(x^2 - 9) = 3x^2 - 27 \rightarrow \boxed{3x^2 \phantom{+ 9x} - 27}$$

deg is not smaller than deg of  $Q(x) = x^2 - 9$ , so keep going

Long division

$$\frac{P(x)}{Q(x)} = \frac{6x^3 + 3x^2 - 45x - 24}{x^2 - 9}$$

$$\text{Goal: } \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

$$\textcircled{1} \quad \frac{6x^3}{x^2} = 6x \quad \textcircled{1'} \quad \frac{3x^2}{x^2} = 3$$

$$6x + 3$$

$$\textcircled{2} \quad 6x(x^2 - 9) = 6x^3 - 54x \rightarrow$$

$$\begin{array}{r} x^2 - 9 \overline{) 6x^3 + 3x^2 - 45x - 24} \\ \underline{6x^3 \phantom{+ 3x^2} - 54x} \phantom{- 24} \\ \phantom{6x^3} 3x^2 + 9x - 24 \end{array}$$

$$\textcircled{3} \quad \text{Subtract with } \begin{array}{r} 6x^3 + 3x^2 - 45x - 24 \\ \underline{6x^3 \phantom{+ 3x^2} - 54x} \\ 3x^2 + 9x - 24 \end{array} \rightarrow$$

$$3x^2 + 9x - 24$$

← deg is not smaller than deg of  $Q(x) = x^2 - 9$ , so keep going

$$\textcircled{2'} \quad 3(x^2 - 9) = 3x^2 - 27 \rightarrow$$

$$3x^2 - 27$$

$$\textcircled{3'} \quad \text{subtract with } \begin{array}{r} 3x^2 + 9x - 24 \\ \underline{3x^2 \phantom{+ 9x} - 27} \\ 9x + 3 \end{array} \rightarrow$$

$$9x + 3$$

deg is smaller than the denominator  $Q(x)$ .  
We're done.

Long division  $\frac{P(x)}{Q(x)} = \frac{6x^3 + 3x^2 - 45x - 24}{x^2 - 9}$

Goal:  $\frac{P(x)}{Q(x)} = \underbrace{S(x)}_{\text{polynomial}} + \underbrace{\frac{R(x)}{Q(x)}}_{\text{proper rational function}}$

$6x + 3$  ← the polynomial  $S(x)$

$$\begin{array}{r} x^2 - 9 \overline{) 6x^3 + 3x^2 - 45x - 24} \\ \underline{6x^3 \phantom{+ 3x^2} - 54x} \phantom{- 24} \\ 3x^2 + 9x - 24 \\ \underline{3x^2 \phantom{+ 9x} - 27} \\ 9x + 3 \end{array}$$

$9x + 3$  ← the remainder  $R(x)$

$$\frac{6x^3 + 3x^2 - 45x - 24}{x^2 - 9} = \boxed{6x + 3} + \frac{\boxed{9x + 3}}{x^2 - 9}$$

polynomial                      proper rational function

Practice w/  
textbook

EXAMPLE 1

$$\frac{x^3 + x}{x - 1}$$



# Partial fraction theorem CASE I

CASE I

If  $\frac{R(x)}{Q(x)}$  is proper and

$Q(x)$  is a product of distinct linear polynomials  
 $(a_1x+b_1)(a_2x+b_2) \dots (a_kx+b_k)$ ,

then


$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \dots + \frac{A_k}{a_kx+b_k} \quad \text{for some numbers } A_1, A_2, \dots, A_k.$$

Example:  $\frac{5}{x^2-4} = \frac{5}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$  for some numbers  $A, B$

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$
 for some numbers  $A, B, C$

$\frac{4x}{(x-1)^2(x+1)}$  does not apply to this CASE I because the factor  $(x-1)$  appears twice! See CASE II

What is the partial fraction decomposition for  $\frac{9x+3}{x^2-9}$  ?

(a)  $\frac{A}{x-3} + \frac{B}{x+3}$    $\frac{9x+3}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}$  by Partial Fraction Theorem CASE I

(b)  $\frac{Ax+B}{x-3} + \frac{Cx+D}{x+3}$

(c)  $\frac{A}{x^2} + \frac{B}{9}$

(d)  $\frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{x+3} + \frac{D}{(x+3)^2}$

# Webwork #1

$$\int \frac{6x^3 + 3x^2 - 45x - 24}{x^2 - 9} dx$$

Step 0 If  $\text{deg}(\text{num}) \geq \text{deg}(\text{denom})$ ,

perform long division:

$$\frac{6x^3 + 3x^2 - 45x - 24}{x^2 - 9} = 6x + 3 + \frac{9x + 3}{x^2 - 9} \quad (\text{see prev.})$$

Step 1 Partial fraction decomposition for  $\frac{9x + 3}{x^2 - 9}$

CASE I: Write  $\frac{9x + 3}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}$

• Multiply both sides by the denominator

$$9x + 3 = A(x+3) + B(x-3)$$

• Write both sides in standard form  
 $a_n x^n + \dots + a_1 x + a_0$

$$9x + 3 = x(A+B) + (3A - 3B)$$

$$\left. \begin{aligned} 9x &= (A+B)x \\ 3 &= 3A - 3B \end{aligned} \right\} \Rightarrow \begin{cases} 9 = A+B \\ 1 = A-B \end{cases}$$

$$\Rightarrow \left. \begin{aligned} 9 &= A+B \\ B &= A-1 \end{aligned} \right\} \begin{aligned} 9 &= A + (A-1) \Rightarrow 10 = 2A \Rightarrow \boxed{A=5} \\ B &= A-1 = 5-1 \Rightarrow \boxed{B=4} \end{aligned}$$

Ready to integrate:

$$\int \frac{6x^3 + 3x^2 - 45x - 24}{x^2 - 9} dx$$

$$= \int \left[ 6x + 3 + \frac{9x + 3}{x^2 - 9} \right] dx$$

$$= \int \left[ 6x + 3 + \frac{5}{x-3} + \frac{4}{x+3} \right] dx$$

$$= \frac{6x^2}{2} + 3x + 5 \ln|x-3| + 4 \ln|x+3| + C$$

Practice w/ textbook examples

**EXAMPLE 1** Find  $\int \frac{x^3 + x}{x-1} dx$

**EXAMPLE 2** Evaluate  $\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} dx$

**EXAMPLE 3** Find  $\int \frac{dx}{x^2 - a^2}$

# Partial fraction theorem CASE II

If  $\frac{R(x)}{Q(x)}$  is proper and

$Q(x)$  has a factor  $(ax+b)$  that is repeated  $r$  times,

instead of  $\frac{A}{ax+b}$ , write  $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \dots + \frac{A_r}{(ax+b)^r}$

Example:  $\frac{x^5}{(x+2)^2 x^4} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x} + \frac{D}{x^2} + \frac{E}{x^3} + \frac{F}{x^4}$

factor  $(x+2)$  is repeated twice       $x$  is repeated four times

$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$

factor  $(x-1)$  is repeated twice

$\frac{x^3 - x + 1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$

$x$  is repeated twice       $x-1$  is repeated three times

What is the partial fraction decomposition for  $\frac{5x^3 - 3x^2 - 8x - 3}{x^4 - 3x^3}$

(a)  $\frac{A}{x^3} + \frac{B}{x-3}$

(b)  $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-3}$



$$\frac{5x^3 - 3x^2 - 8x - 3}{x^3(x-3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-3}$$

because factor  $x$  has 3 repetitions

(c)  $\frac{A}{x} + \frac{B}{x-3}$

(d)  $\frac{A}{x^4} + \frac{B}{3x^3}$

# Webwork #5

$$\int \frac{5x^3 - 3x^2 - 8x - 3}{x^4 - 3x^3} dx$$

CASE II

•  $\deg(\text{numer}) = 3 < 4 = \deg(\text{denom})$ .

Don't need to do long division.

• Partial fraction decomposition CASE II

$$\frac{5x^3 - 3x^2 - 8x - 3}{x^3(x-3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-3}$$

*x has 3 repetitions*

$$\begin{aligned} 5x^3 - 3x^2 - 8x - 3 &= Ax^2(x-3) + Bx(x-3) + C(x-3) + Dx^3 \\ &= Ax^3 - 3Ax^2 + Bx^2 - 3Bx + Cx - 3C + Dx^3 \\ &= x^3[A+D] + x^2[-3A+B] + x[-3B+C] - 3C \end{aligned}$$

$$\begin{array}{l} 5 = A + D \\ -3 = -3A + B \\ -8 = -3B + C \\ -3 = -3C \end{array} \quad \left. \begin{array}{l} 5 = 2 + D \Rightarrow D = 3 \\ -3 = -3A + 3 \Rightarrow A = 2 \\ -8 = -3B + 1 \Rightarrow B = 3 \\ C = 1 \end{array} \right\}$$

Ready to integrate:

$$\int \frac{5x^3 - 3x^2 - 8x - 3}{x^3(x-3)} dx$$

$$= \int \left[ \frac{2}{x} + \frac{3}{x^2} + \frac{1}{x^3} + \frac{3}{x-3} \right] dx$$

$$= \left( 2 \ln|x| - \frac{3}{x} - \frac{1}{2x^2} + 3 \ln|x-3| \right) + C$$

$$= \ln\left(x^2|(x-3)^3|\right) - \frac{3}{x} - \frac{1}{2x^2} + C$$

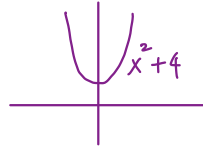
Practice w/ textbook example

**EXAMPLE 4** Find  $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} dx$

## Partial fraction theorem CASE III

A quadratic polynomial  $ax^2 + bx + c$  has no real root iff  $b^2 - 4ac < 0$ ,

e.g.  $x^2 + 4$



If  $Q(x)$  has a quadratic factor  $ax^2 + bx + c$

which has no real root, then

the partial fraction decomposition of  $\frac{R(x)}{Q(x)}$  has a term  $\frac{Ax + B}{ax^2 + bx + c}$

Example :

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4}$$

$$\frac{2x^2 - x + 4}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

## Partial fraction theorem CASE IV

If  $Q(x)$  has a quadratic factor  $ax^2 + bx + c$  (with no real root)

which is repeated  $r$  times,

the partial fraction decomposition of  $\frac{R(x)}{Q(x)}$  has the sum

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \dots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

Examples: •  $\frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2} = \frac{A}{x} + \underbrace{\frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}}_{\text{two repeats}}$

•  $\frac{x^3 + x^2 + 1}{x(x - 1)(x^2 + x + 1)(x^2 + 1)^3} = \frac{A}{x} + \frac{B}{x - 1} + \frac{Cx + D}{x^2 + x + 1} + \underbrace{\frac{Ex + F}{x^2 + 1} + \frac{Gx + H}{(x^2 + 1)^2} + \frac{Ix + J}{(x^2 + 1)^3}}_{\text{three repeats}}$



# Rationalizing substitutions (Sec 7.4)

Turn a non-rational integrand into a rational integrand by substitution.

$$\text{Webwork \#8: } \int \frac{1}{x} \frac{1}{\sqrt{x+1}} dx = \int \frac{1}{(u^2-1)} \frac{1}{u} 2u du = \int \frac{2}{u^2-1} du = \int \frac{2}{(u-1)(u+1)} du$$

$$u = \sqrt{x+1}$$

$$u^2 = x+1$$

$$x = u^2 - 1$$

$$dx = 2u du$$

Alternatively:

$$du = \frac{1}{2} \frac{1}{\sqrt{x+1}} dx$$

$$du = \frac{1}{2} \frac{1}{u} dx$$

$$2u du = dx$$

Partial fraction decomposition case I:

$$\frac{2}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1}$$

$$\text{Webwork \#6: } \int \frac{34e^x + 80}{e^{2x} + 7e^x + 10} dx = \int \frac{34u + 80}{u^2 + 7u + 10} \frac{1}{u} du = \int \frac{34u + 80}{u(u+2)(u+5)} du$$

$$u = e^x$$

$$\ln u = x$$

$$dx = \frac{1}{u} du$$

Alternatively:

$$du = e^x dx$$

$$du = u dx$$

$$\frac{1}{u} du = dx$$

Partial fraction decomposition case I:

$$\frac{34u + 80}{u(u+2)(u+5)} = \frac{A}{u} + \frac{B}{u+2} + \frac{C}{u+5}$$