Sec 7.2 Trig Integrals Strategies

Memorize $1 = \sin^2 x + \cos^2 x$ $\quad \dfrac{1}{\cos^2 x} = \dfrac{\sin^2 x}{\cos^2 x} + \dfrac{\cos^2 x}{\cos^2 x} \iff (\sec x)^2 = (\tan x)^2 + 1$

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product of sin & cos w/ "odd cos"

1. Save $(\cos x)$ because $\frac{d}{dx} \sin x = \cos x$
2. Turn all other $\cos x$ into $\sin x$ by $1 = \sin^2 x + \cos^2 x \iff \cos^2 x = 1 - \sin^2 x$
3. Sub $u = \sin x$, $du = \cos x \, dx$

(Same strategy for "odd sin", but swap sin & cos)

Even cos or sin

Apply $\cos^2 x = \dfrac{1 + \cos 2x}{2}$ or $\sin^2 x = \dfrac{1 - \cos 2x}{2}$

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product of sec & tan with...

"even sec":	"odd tan"
1. Save $(\sec x)^2$ because $\frac{d}{dx} \tan x = (\sec x)^2$	1. Save $\sec x \tan x$ since $\frac{d}{dx} \sec x = \sec x \tan x$
2. Turn all other $(\sec x)^2$ into $1 + (\tan x)^2$	2. Turn all $(\tan x)^2$ into $\sec^2 x - 1$
3. Sub $u = \tan x$, $du = (\sec x)^2 \, dx$	3. Sub $u = \sec x$, $du = \sec x \tan x \, dx$

Odd powers of cosine

"odd cos"

$$\int (\cos x)^7\, dx = \int (\cos x)^6 \cos x\, dx$$

• Put one $\cos x$ aside

$$= \int \left[(\cos x)^2\right]^3 \cos x\, dx$$

• convert $(\cos x)^{2k}$ to $\left[(\cos x)^2\right]^k$

$$= \int \left[1 - (\sin x)^2\right]^3 \cos x\, dx$$

• Use $1 = \cos^2 x + \sin^2 x$

• Apply u-sub

Let $u = \sin x$

$du = \cos x\, dx$

$$= \int \left[1 - u^2\right]^3 du$$

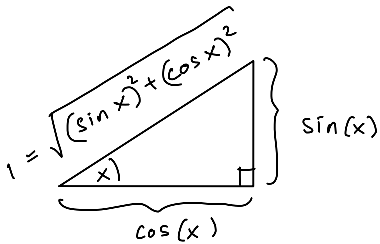
• Multiply out

$$= \int (1-u^2)(1-u^2)(1-u^2)\, du$$

$$= \int \left[1 - 3u^2 + 3u^4 - u^6\right] du$$

$$= u - 3\frac{u^3}{3} + 3\frac{u^5}{5} - \frac{u^7}{7} + C$$

$$= \boxed{(\sin x) - (\sin x)^3 + \frac{3}{5}(\sin x)^5 - \frac{(\sin x)^7}{7} + C}$$



MEMORIZE

$$\boxed{1 = (\cos(x))^2 + (\sin(x))^2}$$

Practice: $\int (\cos x)^3\, dx$

Textbook Example 1

Odd powers of ~~cosine~~ sine

"odd sin"

$$\int (\sin x)^7\,dx = \int (\sin x)^6 \sin x\,dx$$

~~Put one cos x aside~~

- Put one $\sin x$ aside

$$= \int \left[(\sin x)^2\right]^3 \sin x\,dx$$

- convert $(\sin x)^{2k}$ to $\left[(\sin x)^2\right]^k$

(The same strategy if we replace cosine with sine)

$$= \int \left[1-(\cos x)^2\right]^3 \sin x\,dx$$

- Use $1=\cos^2 x+\sin^2 x$

$$= \int -\left[1-u^2\right]^3\,du$$

- Let $u = \cos x$

  $du = -\sin x\,dx$

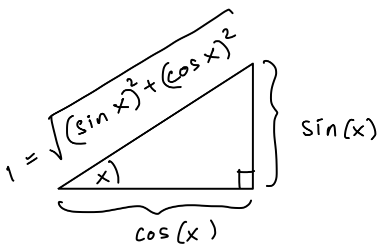
$$= \int -(1-u^2)(1-u^2)(1-u^2)\,du$$

- Multiply out

$$= -\int \left[1-3u^2+3u^4-u^6\right]du$$

$$= -u + 3\frac{u^3}{3} - 3\frac{u^5}{5} + \frac{u^7}{7} + C$$

$$= -(\cos x) + (\cos x)^3 - \frac{3}{5}(\cos x)^5 + \frac{(\cos x)^7}{7} + C$$



MEMORIZE

$$\boxed{1 = (\cos(x))^2 + (\sin(x))^2}$$

practice method

- Textbook Example 1, 2
- Pre-class reading KA

Product of sine and cosine; Powers are both even

"even sin"

(Appendix D, pg A 29): $(\cos x)^2 = \dfrac{1 + \cos(2x)}{2}$

$(\sin x)^2 = \dfrac{1 - \cos(2x)}{2}$

Will be given on exams, but useful to memorize

If both sin & cos are there, may need

$\sin x \cos x = \frac{1}{2} \sin 2x$

$$\int (\sin x)^4 \, dx = \int \left[(\sin x)^2\right]^2 dx$$

$$= \int \left[\frac{1 - \cos(2x)}{2}\right]^2 dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos(2x) + \left[\cos(2x)\right]^2\right) dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos(2x) + \left[\frac{1 + \cos(4x)}{2}\right]\right) dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos(2x) + \frac{1}{2}\cos(4x)\right) dx$$

$$= \frac{1}{4}\left[\frac{3}{2}x - \frac{2}{2}\sin(2x) + \frac{1}{2}\,\frac{\sin(4x)}{4}\right] + C$$

practice method

Textbook Example 3,4

Webwork Problem 8

"even sin"

$$\int_0^{\frac{\pi}{4}} (\cos x)^2 (\tan x)^2 \, dx$$

$$\int (\cos x)^2 (\tan x)^2 \, dx = \int (\sin x)^2 \, dx$$

$$= \int \frac{1}{2}(1 - \cos(2x)) \, dx$$

$$= \frac{1}{2}\left[x - \frac{\sin(2x)}{2}\right] + C$$

$$= \frac{x}{2} - \frac{\sin(2x)}{4} + C$$

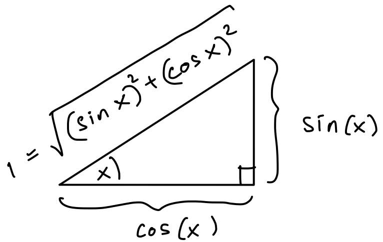
$$\int_0^{\frac{\pi}{4}} (\cos x)^2 (\tan x)^2 \, dx = \frac{x}{2} - \frac{\sin(2x)}{4} \Bigg|_0^{\frac{\pi}{4}}$$

$$= \left[\frac{\pi}{8} - \frac{\sin\left(\frac{\pi}{2}\right)}{4}\right] - \left[0 - \frac{\sin(0)}{4}\right]$$

$$= \boxed{\frac{\pi}{8} - \frac{1}{4}}$$

Product of tangent & secant ; Power of secant is even

"even sec"



MEMORIZE

$$1 = (\cos(x))^2 + (\sin(x))^2$$

$$\underbrace{1}_{(\cos x)^2} = \underbrace{(\cos(x))^2}_{(\cos x)^2} + \underbrace{(\sin(x))^2}_{(\cos x)^2}$$

$$\boxed{(\sec x)^2 = 1 + (\tan x)^2}$$

- $\int (\sec x)^2 \, dx = \boxed{\tan x + C}$ because $\frac{d}{dx} \boxed{\tan(x)} = [\sec(x)]^2$

- $\int (\sec x)^2 (\tan x)^2 \, dx = \int u^2 \, du = \frac{u^3}{3} + C = \boxed{\frac{(\tan x)^3}{3} + C}$

$u = \tan x$
$du = (\sec x)^2 \, dx$

- $\int (\tan x)^4 \, dx = \int (\tan x)^2 (\tan x)^2 \, dx$

$$= \int [(\sec x)^2 - 1] (\tan x)^2 \, dx$$

Apply $(\tan x)^2 = (\sec x)^2 - 1$

$$= \underbrace{\int (\sec x)^2 (\tan x)^2 dx}_{\text{computed above}} - \underbrace{\int (\tan x)^2 \, dx}_{\text{Repeat the same process}}$$

$$= \frac{(\tan x)^3}{3} - \int [(\sec x)^2 - 1] \, dx$$

$$= \boxed{\frac{(\tan x)^3}{3} - (\tan x) + x + C}$$

Practice

Textbook Example 5, 7

Product of tangent & secant ; Power of tangent is odd

"odd tan"

$$\int_0^{\frac{\pi}{4}} (\tan x)^3 (\sec x)^3 \, dx = \int_0^{\frac{\pi}{4}} (\tan x)^2 (\sec x)^2 \underbrace{\sec(x)\tan(x)} \, dx$$

put one tan(x) sec(x) aside

because $\frac{d}{dx} \sec(x) = \sec x \tan x$

$$= \int_0^{\frac{\pi}{4}} \left[(\sec x)^2 - 1\right] (\sec x)^2 \sec(x)\tan(x) \, dx$$

Apply $(\tan x)^2 = (\sec x)^2 - 1$

$$= \int_{u = \sec 0}^{u = \sec \frac{\pi}{4}} \left[u^2 - 1\right] u^2 \, du$$

$u = \sec x$

$du = \sec x \tan x \, dx$

$$= \int_{\frac{1}{\cos 0} = 1}^{\frac{1}{\cos \frac{\pi}{4}} = \frac{2}{\sqrt{2}} = \sqrt{2}} (u^4 - u^2) \, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} \Bigg|_{u=1}^{u=\sqrt{2}}$$

$$= \boxed{\frac{1}{5} 2^{\frac{5}{2}} - \frac{1}{5} - \frac{1}{3} 2^{\frac{3}{2}} + \frac{1}{3}}$$

$$\frac{1}{(\cos x)^2} = \frac{(\cos(x))^2}{(\cos x)^2} + \frac{(\sin(x))^2}{(\cos x)^2}$$

$$\boxed{(\sec x)^2 = 1 + (\tan x)^2}$$

Similar situations

- Textbook Example 6
- Next Section Webwork 7.3 #3

The above guidelines are clear-cut. Other cases require different methods.

Review Sec 6.4

- $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$

  Apply u-sub and $\int \frac{1}{u} \, du = \ln|u|$

  $u = \cos x$

  $du = -\sin x \, dx$

- $\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx$

  Apply u-sub and $\int \frac{1}{u} \, du = \ln|u|$

  $u = \sin x$

  $du = \cos x \, dx$

$$\int \sec(x)\, dx$$

$$\int \sec(x)\, dx = \int \sec(x) \frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\, dx$$

because

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$

$$\frac{d}{dx}\tan(x) = [\sec(x)]^2$$

$$= \int \frac{[\sec(x)]^2 + \sec(x)\tan(x)}{\sec(x)+\tan(x)}\, dx$$

$$u = \sec x + \tan x$$

$$du = [\sec x \tan x + (\sec x)^2]\, dx$$

$$= \int \frac{1}{u}\, du$$

$$= \ln|u| + C$$

$$= \ln|\sec x + \tan x| + C$$

— the end —