Sec 7.1 Integration by Parts

Differentiation		Integration
Chain rule	$\longleftrightarrow$	u-substitution
product rule	$\longleftrightarrow$	Integration by parts

$$u v' + v u' = (uv)'$$

$$\int u \, dv + \int v \, du = u v$$

$$\boxed{\int u \, dv = u v - \int v \, du}$$

Example #1 $\int x \cos(x)\, dx = ?$

$$\boxed{\int u\, dv = uv - \int v\, du}$$

Think:

- Can we do u-substitution?

  x and $\cos(x)$ are "unrelated" by differentiation, so u-sub won't work

- Try Integration by Parts

Pick

u	dv
du	v

so that: $\int v\, du$ is simpler than (or at least not more complicated) the original integral

If I pick

$u = \cos(x)$	$dv = x\, dx$
$du = -\sin(x)$	$v = \frac{x^2}{2}$

then $\int v\, du = \underbrace{\int \frac{x^2}{2}(-\sin(x))\, dx}$

more complicated than the original $\int x \cos(x)\, dx$

So I try

$u = x$	$dv = \cos x\, dx$
$du = dx$	$v = \sin x$

then $\int v\, du = \underbrace{\int \sin(x)\, dx}$

we know how to solve :)

(con't to next page)

Example #1 $\int x \cos(x)\, dx = ?$

$$\boxed{\int u\, dv = uv - \int v\, du}$$

Pick

$u = x$	$dv = \cos x\, dx$
$du = dx$	$v = \sin x$

$$\int x \cos(x)\, dx = uv - \int v\, du$$

$$= x \sin x - \int \sin x\, dx$$

$$= \boxed{x \sin x + \cos x + C}$$

Practice similar problems

- Textbook Example 1
- Recommended KA

Example #2 Evaluate $\int e^x \sin x \, dx$

Think: u-substitution does not work because e^x and $\sin x$ are "unrelated".
So try Integration by Parts.

Things to try

a) Apply Integration by Parts with

$u = \sin x$	$dv = e^x \, dx$
$du = \cos x \, dx$	$v = e^x$

b) Apply Integration by Parts with

$u = e^x$	$dv = \sin(x) \, dx$
$du = e^x \, dx$	$v = -\cos(x)$

⇦ Textbook Example 4 uses option (b), so we'll do option (a) so you can see both options

c) Integration by Parts won't work?

Example#2 Evaluate $\int e^x \sin x \, dx$

Try

$u = \sin x$	$dv = e^x \, dx$
$du = \cos x \, dx$	$v = e^x$

• $\int v \, du = \int e^x \cos x \, dx$ is not simpler, but it's not more complicated, so it's OK.

$$\int e^x \sin x \, dx = uv - \int v \, du$$

$$= (\sin x) e^x - \int e^x \cos x \, dx$$

$u = \cos x$	$dv = e^x \, dx$
$du = -\sin x \, dx$	$v = e^x$

$$= (\sin x) e^x - \left[uv - \int v \, du \right]$$

$$= (\sin x) e^x - \left[(\cos x) e^x - \int e^x (-\sin x) \, dx \right]$$

$$= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

$$\int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

$$2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x$$

$$\int e^x \sin x \, dx = \frac{1}{2} e^x \left[\sin x - \cos x \right] + C$$

Practice:

Try $u = e^x$, $dv = \sin x \, dx$

See solution in

- Textbook Example 4
- Recommended KA

Example #3

Evaluate $\int \arcsin(x)\,dx$

The only possible choice

$u = \arcsin(x)$	$dv = dx$
$du = \dfrac{1}{\sqrt{1-x^2}}\,dx$	$v = x$

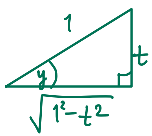
$y = \arcsin(t)$

$\sin(y) = t$

↓ Implicit Differentiation

$\cos(y)\dfrac{dy}{dt} = 1$

$\dfrac{dy}{dt} = \dfrac{1}{\cos(y)}$



$\dfrac{d}{dt}\left(\arcsin(t)\right) = \dfrac{1}{\sqrt{1-t^2}}$

Example#3 Evaluate $\int \arcsin(x)\, dx$

$u = \arcsin(x)$	$dv = dx$
$du = \frac{1}{\sqrt{1-x^2}}dx$	$v = x$

Similar to Webwork

$$\int \arcsin(x)\, dx = uv - \int v\, du$$

$$= x \arcsin(x) - \underbrace{\int x \frac{1}{\sqrt{1-x^2}}\, dx}$$

$w = 1-x^2 \quad dw = -2x\, dx$

$-\frac{1}{2}dw = x\, dx$

$$= x \arcsin(x) - \int \frac{1}{\sqrt{w}}\left(-\frac{1}{2}\right) dw$$

$$= x \arcsin(x) + \int \frac{1}{2} w^{-\frac{1}{2}}\, dw$$

$$= x \arcsin(x) + \cancel{\frac{1}{2}} \frac{w^{\frac{1}{2}}}{\cancel{\left(\frac{1}{2}\right)}} + C$$

$$= \boxed{x \arcsin(x) + \sqrt{1-x^2} + C}$$

Example #36 Evaluate $\int_0^1 \arcsin(x)\, dx$

Recall:

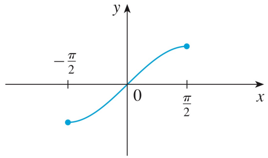
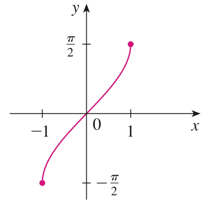
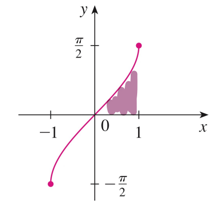


FIGURE 2
$y = \sin x,\ -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$



so $\int_0^1 \arcsin(x)$ is the area under the curve



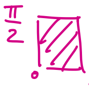
$$\int \arcsin(x)\, dx = x \arcsin(x) + \sqrt{1-x^2} + C$$ (from prev. page)

$$\int_0^1 \arcsin(x)\, dx = x \arcsin(x) \Big|_0^1 + \sqrt{1-x^2}\Big|_0^1$$

$$= 1 \arcsin(1) - \underbrace{0 \cdot \arcsin(0)}_{0} + \left(\underbrace{\sqrt{1-1^2}}_{0} - \sqrt{1-0^2}\right)$$

$$= \frac{\pi}{2} - 1$$

Sanity check: The area under the curve should be

- positive $\frac{\pi}{2} - 1 \approx \frac{3.14}{2} - 1 > 0$ ✓
- smaller than (the area of $\frac{\pi}{2}$ ) $= \frac{\pi}{2}$ ✓

Practice with definite integrals

- Textbook, Example 5
- Recommended KA with definite integrals

Students' questions from class:

Webwork 7.1
Prob 7

Suppose $f(1) = -4$,
$f(4) = -9$,
$f'(1) = 7$,
$f'(4) = 8$,

and f'' is continuous.

Find $\int_1^4 x f''(x)\,dx$

$$\int_1^4 x f''(x)\,dx$$

$u = x$	$dv = f''(x)\,dx$
$du = dx$	$v = f'(x)$

$$= uv\Big|_1^4 - \int_1^4 v\,du$$

$$= x f'(x)\Big|_1^4 - \int_1^4 f'(x)\,dx$$

$$= x f'(x)\Big|_1^4 - \left[f(4) - f(1)\right]$$

$$= 4 \cdot \underbrace{f'(4)}_{8} - 1 \cdot \underbrace{f'(1)}_{8} - \left[\underbrace{f(4)}_{-9} - \underbrace{f(1)}_{-4}\right]$$

Students' questions from class

WW 6.6 Prob 1

$$y = \arctan\left(\sqrt{5x^2 - 1}\right) \qquad \frac{dy}{dx} = ?$$

$$\frac{dy}{dx} = \frac{1}{1+(5x^2-1)} \cdot \left(\frac{1}{2}(5x^2-1)^{\frac{1}{2}-1} \cdot (10x)\right)$$

$$\left(\text{because } \frac{d}{dx}\left[\tan^{-1}(x)\right] = \frac{1}{1+x^2}\right)$$

$$= \frac{1}{5x^2} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{5x^2-1}} \; 10x$$

$$= \frac{1}{10} \cdot \frac{1}{x^2} \cdot \frac{1}{\sqrt{5x^2-1}} \; 10x$$

$$= \boxed{\frac{1}{x} \; \frac{1}{\sqrt{5x^2-1}}}$$