Section 6.8 Indeterminate Forms
l'Hospital's Rule

Read before / after class:

Textbook Examples

1, 5, 6, 9

"$\frac{0}{0}$"

compute directly

"$0 \cdot \infty$"

"indeterminate power"

0^0 or ∞^0 or 1^∞

Strategies:

1. "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$" $\Rightarrow$ l'Hospital's Rule

2. "$\infty - \infty$", "$0 \cdot \infty$" $\Rightarrow$ Turn into "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$"

3. "0^0", "∞^0", "1^∞" $\Rightarrow$ Apply $\ln(x)$ to get "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$"

Review Sec 1.6 Limit Laws

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)$$

exist. Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$
4. $\lim_{x \to a} [f(x)\, g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$

Law #5 doesn't work when the limit of the denominator is 0

Example

$$\boxed{\lim_{x \to \left(\frac{\pi}{2}\right)^-} \frac{1 - \sin x}{\cos x}} = ?$$

- $\lim_{x \to \left(\frac{\pi}{2}\right)^-} 1 - \sin x = \boxed{1 - \sin\left(\frac{\pi}{2}\right) = 1 - 1 = 0}$
- $\lim_{x \to \left(\frac{\pi}{2}\right)^-} \cos x = \boxed{\cos \frac{\pi}{2} = 0}$

called an indeterminate form of type "$\frac{0}{0}$"

- If $\underbrace{(1 - \sin x)}_{\text{numerator}}$ goes to 0 faster, the answer is 0
- If $\underbrace{\cos x}_{\text{denominator}}$ goes to 0 faster, the answer is ∞
- If there is a compromise, the answer is a nonzero number

l'Hospital's Rule "$\frac{0}{0}$"

Rule "$\frac{0}{0}$"

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0.$$

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

The rule works if we replace $\boxed{\lim_{x\to a}}$ with

$\lim_{x\to a^+}$ $\quad \lim_{x\to a^-}$ $\quad \lim_{x\to \infty}$ or $\lim_{x\to -\infty}$

Example: Evaluate $\displaystyle\lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{1-\sin x}{\cos x}$

Solution:

- $\displaystyle\lim_{x\to\left(\frac{\pi}{2}\right)^-} 1-\sin x = 0$
- $\displaystyle\lim_{x\to\left(\frac{\pi}{2}\right)^-} \cos x = 0$

so we can attempt to apply l'Hospital's Rule "$\frac{0}{0}$"

Here $f(x) = 1-\sin x$

$g(x) = \cos x$

$$\lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{1-\sin x}{\cos x} = \lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{\frac{d}{dx}(1-\sin x)}{\frac{d}{dx}(\cos x)} = \lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{-\cos x}{-\sin x} = \frac{\lim_{x\to\left(\frac{\pi}{2}\right)^-} \cos x}{\lim_{x\to\left(\frac{\pi}{2}\right)^-} \sin x} = \frac{\cos\left(\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2}\right)} = \frac{0}{1} = \boxed{0}$$

By l'Hospital's Rule "$\frac{0}{0}$"

Practice l'Hospital's Rule "$\frac{0}{0}$":

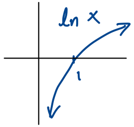
Textbook Example 1, 4

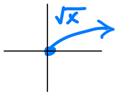
Webwork 5, 6

Recommended Khan Academy "$\frac{0}{0}$" Practice

l'Hospital's Rule "$\frac{\infty}{\infty}$"

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \ ?$$





$$\boxed{\begin{aligned} &\lim_{x\to\infty} \ln x = \infty \\ &\lim_{x\to\infty} \sqrt{x} = \infty \end{aligned}}$$

- If $\ln x$ wins, ? is $\boxed{\infty}$
- If $\sqrt{x}$ wins, ? is $\boxed{0}$
- If there is a compromise, ? is $\boxed{\text{a positive number}}$

In Sec 6.3, we said $\ln x$ "grows slower than any power function, so $\sqrt{x}$ should win.

Rule "$\frac{\infty}{\infty}$"

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to a} g(x) = \pm\infty$$

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

The rule works if we replace $\boxed{\lim_{x\to a}}$ with

$\lim_{x\to a^+}$ $\quad \lim_{x\to a^-}$ $\quad \lim_{x\to \infty}$ or $\lim_{x\to -\infty}$

- $\lim_{x\to a} \frac{f(x)}{g(x)}$ is called an indeterminate form of type "$\frac{\infty}{\infty}$"

Next

Apply this l'Hospital's Rule "$\frac{\infty}{\infty}$" to the above situation with

$$f(x) = \ln x$$
$$g(x) = \sqrt{x}$$

l'Hospital's Rule "$\frac{\infty}{\infty}$"

Example: $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = ?$

Solution: $\left.\begin{array}{l} \lim_{x\to\infty} \ln x = \infty \\ \lim_{x\to\infty} \sqrt{x} = \infty \end{array}\right\}$ so we can attempt to apply l'Hospital's Rule "$\frac{\infty}{\infty}$"

(from prev. slide)

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}\left(x^{\frac{1}{2}}\right)} = \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{2}x^{-\frac{1}{2}}\right)} = \lim_{x\to\infty} \frac{1}{x} \, 2x^{\frac{1}{2}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = \boxed{0}$$

due to l'Hospital's Rule "$\frac{\infty}{\infty}$"

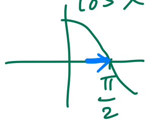
So $\sqrt{x}$ wins,
it grows much faster than $\ln x$,
as expected
because power funs
grow faster than $\ln(x)$

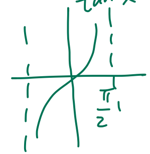
Indeterminate Differences

type "$\infty - \infty$"

EXAMPLE 8 Compute $\lim_{x\to(\pi/2)^-}(\sec x - \tan x)$

$\frac{1}{\cos x}$





- $\lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{1}{\cos x} = +\infty$ and $\lim_{x\to\left(\frac{\pi}{2}\right)^-} \tan x = +\infty$

- If $\sec(x)$ wins, the answer is ∞
- If $\tan(x)$ wins, the answer is $-\infty$
- If they compromise, the answer is a number

Strategy: Convert this difference into a quotient $\frac{\text{a function}}{\text{another function}}$

$$\sec(x) - \tan(x) = \frac{1}{\cos(x)} - \frac{\sin(x)}{\cos(x)} = \frac{1-\sin(x)}{\cos(x)}$$

$$\left.\begin{aligned} &\lim_{x\to\left(\frac{\pi}{2}\right)^-} 1-\sin(x) = 0 \\ &\lim_{x\to\left(\frac{\pi}{2}\right)^-} \cos(x) = 0 \end{aligned}\right\}$$

so we can try to use L'Hospital's Rule "$\frac{0}{0}$"

- we already computed $\lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{1-\sin x}{\cos x}$

Indeterminate Differences

type "$\infty - \infty$"

General Strategy

If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$

and we want to compute $\lim_{x \to a} f(x) - g(x) \ldots$

EXAMPLE 8 Compute $\lim_{x \to (\pi/2)^-} (\sec x - \tan x)$

$$\lim_{x \to (\pi/2)^-} (\sec x - \tan x) = \lim_{x \to (\pi/2)^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right)$$

$$= \lim_{x \to (\pi/2)^-} \frac{1 - \sin x}{\cos x} = \lim_{x \to (\pi/2)^-} \frac{-\cos x}{-\sin x} = 0$$

First, turn into a quotient

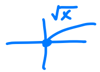
Next, try to apply l'H Rule "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$"

Note that the use of l'Hospital's Rule is justified because $1 - \sin x \to 0$ and $\cos x \to 0$ as $x \to (\pi/2)^-$. ■

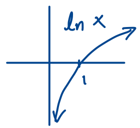
Webwork Prob 7 : Evaluate $\lim_{x\to 0^+} \sqrt{x}\,\ln(x)$

type "$0 \cdot \infty$"

$\lim_{x\to 0^+} \sqrt{x} = 0$



$\lim_{x\to 0^+} \ln(x) = -\infty$



- If $\sqrt{x}$ wins, the answer is 0
- If $\ln(x)$ wins, the answer is $-\infty$
- If there is a compromise, the answer is a nonzero number

Turn $\sqrt{x}\,\ln(x)$ into a quotient by writing $\boxed{\dfrac{\sqrt{x}}{\left(\frac{1}{\ln x}\right)}}$ or $\dfrac{\ln(x)}{\left(\frac{1}{\sqrt{x}}\right)}$

First, try writing $\dfrac{\sqrt{x}}{\left(\frac{1}{\ln x}\right)}$

$\lim_{x\to 0^+} \sqrt{x} = 0$

$\lim_{x\to 0^+} \dfrac{1}{\ln x} = 0$

So we can attempt to use L'Hospital's Rule "$\frac{0}{0}$"

WARNING

The following statement is **FALSE**

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$,

then $\lim_{x\to a} f(x)\,g(x) = 0$

NOT A TRUE STATEMENT

FALSE!

$$\frac{\sqrt{x}}{\left(1/\ln x\right)} \qquad \frac{\frac{d}{dx}\sqrt{x}}{\frac{d}{dx}\left(\frac{1}{\ln x}\right)} \qquad \frac{\frac{1}{2}x^{-\frac{1}{2}}}{-\frac{1}{(\ln x)^2}\cdot\frac{1}{x}} \qquad -\frac{1}{2}\sqrt{x}\,(\ln x)^2$$

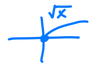
More complicated than the expression we started with.

Next: try the other quotient

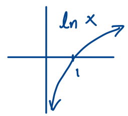
Webwork Prob 7 : Evaluate $\lim_{x \to 0^+} \sqrt{x} \ln(x)$

type "$0 \cdot \infty$"

$\lim_{x \to 0^+} \sqrt{x} = 0$



$\lim_{x \to 0^+} \ln(x) = -\infty$



Turn $\sqrt{x} \ln(x)$ into a quotient by writing $\dfrac{\sqrt{x}}{\left(\frac{1}{\ln x}\right)}$ or $\dfrac{\ln(x)}{\left(\frac{1}{\sqrt{x}}\right)}$

So try writing as $\dfrac{\ln(x)}{\left(\frac{1}{\sqrt{x}}\right)}$

$$\left.\begin{aligned} \lim_{x \to 0^+} \ln x &= -\infty \\ \lim_{x \to 0^+} \frac{1}{\sqrt{x}} &= \infty \end{aligned}\right\}$$

So we can attempt to use L'Hospital's Rule "$\frac{\infty}{\infty}$"

$$\lim_{x \to 0^+} \frac{\ln(x)}{\left(x^{-\frac{1}{2}}\right)} = \lim_{x \to 0^+} \frac{\frac{d}{dx} \ln(x)}{\frac{d}{dx}\left(x^{-\frac{1}{2}}\right)} = \lim_{x \to 0^+} \frac{\left(\frac{1}{x}\right)}{-\frac{1}{2} x^{-\frac{3}{2}}} = \lim_{x \to 0^+} \frac{x^{-1}}{\left(-\frac{1}{2}\right) x^{-\frac{3}{2}}} = \lim_{x \to 0^+} -2\, x^{\frac{3}{2}-1} = \lim_{x \to 0^+} -2 x^{\frac{1}{2}} = \boxed{0}$$

(so $\sqrt{x}$ wins!)

Practice other "Indeterminate Products"

- Textbook, Example 6
- Textbook, Example 7
- Webwork 9

EXAMPLE 5 Find $\displaystyle\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x}$.

Not an indeterminate form

EXAMPLE 5 Find $\displaystyle\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x}$.

Solution:

$$\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x} = \frac{\sin \pi}{1 - \cos \pi} = \frac{0}{1 - (-1)} = 0$$

- $\sin(x)$ is continuous at $x = \pi$
- $1 - \cos(x)$ is continuous at $x = \pi$

Type "0^0", "∞^0", "1^∞"

Indeterminate Powers

Several indeterminate forms arise from the limit

$$\lim_{x \to a} [f(x)]^{g(x)}$$

1. $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ type 0^0
2. $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = 0$ type ∞^0
3. $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \pm\infty$ type 1^∞

Strategy for all:
Apply $\ln$

$$y = [f(x)]^{(g(x))}$$

$$\ln y = \ln\left[(f(x))^{g(x)}\right]$$

$$\ln y = g(x) \ln(f(x))$$

$\lim_{x \to a} \ln y$ will now be an indeterminate form of type "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$"

Indeterminate power Example

Example:

$$\lim_{x\to 0^+}(1+x)^{1/x} = ?$$

Answer:

$$\left.\begin{aligned}\lim_{x\to 0^+}(1+x) &= \boxed{1}\\ \lim_{x\to 0^+}\frac{1}{x} &= \boxed{\infty}\end{aligned}\right\} \text{ Type } \boxed{\text{"}1^\infty\text{"}}$$

$$y=(1+x)^{\frac{1}{x}}$$

$$\ln y = \ln\left((1+x)^{\frac{1}{x}}\right)$$

$$\ln y = \underbrace{\frac{1}{x}\ln(1+x)}$$

Now, work with $\dfrac{\ln(1+x)}{x}$

$$\left.\begin{aligned}\lim_{x\to 0^+}\ln(1+x) &= 0\\ \lim_{x\to 0^+} x &= 0\end{aligned}\right\} \text{ Try l'H Rule}$$

$$\lim_{x\to 0^+}\frac{\ln(1+x)}{x} = \lim_{x\to 0^+}\frac{\frac{d}{dx}[\ln(1+x)]}{\frac{d}{dx}(x)} = \lim_{x\to 0^+}\frac{\left(\frac{1}{1+x}\right)}{1} = 1$$

by l'H Rule "$\frac{0}{0}$"

$$\text{so } \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{(\ln y)} = e^{\left(\lim_{x\to 0^+}\ln y\right)} = e^1 = \boxed{e}$$

Practice similar question: $\lim_{x\to\infty}\left(\frac{1}{x}+1\right)^x = ?$

Hint: The answer is NOT 1

Try $\lim_{x\to 0^+} x^x = ?$ See the last example in the textbook

Strategies

1. Try to compute the limit directly

2. If it doesn't work, try to write as a quotient $\frac{f(x)}{g(x)}$

   * If $\left\{ \begin{array}{l} \lim_{x\to a} f(x) = 0, \text{ and} \\ \lim_{x\to a} g(x) = 0 \end{array} \right\}$ OR if $\left\{ \begin{array}{l} \lim_{x\to a} f(x) = \infty, \text{ and} \\ \lim_{x\to a} g(x) = \infty \end{array} \right\}$,

   try l'Hospital's Rule

3. If $\lim_{x\to a} [f(x)]^{g(x)}$ is of type "0^0", "∞^0", or "1^∞"

   write $y = f(x)^{g(x)}$

   $\ln y = g(x) \ln(f(x))$

   Compute $\lim_{x\to a} \ln y$ first. Then $\lim_{x\to a} y = e^{\left(\lim_{x\to a} \ln y\right)}$