Sec 6.6 Inverse Trig Functions

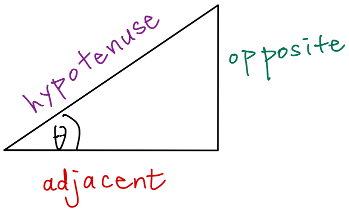
$\boxed{\text{Part a: Trigonometry Review}}$

Part b: Calculus of arcsin (x) and arccos (x)

Part c: Calculus of other inverse trig functions

Sec 6.6 part (a) Trigonometry

[Go to Appendix D Trigonometry pg A26 eq $\boxed{4}$]



Def When θ ("theta") is in $(0, \frac{\pi}{2})$ ...

$$\sin\theta \overset{def}{=} \frac{\text{opposite}}{\text{hypotenuse}} \qquad \csc\theta \overset{def}{=} \frac{1}{\sin\theta}$$

$$\cos\theta \overset{def}{=} \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sec\theta \overset{def}{=} \boxed{\frac{1}{\cos\theta}}$$

$$\tan\theta \overset{def}{=} \boxed{\frac{\text{opposite}}{\text{adjacent}}} \qquad \cot\theta \overset{def}{=} \boxed{\frac{1}{\tan\theta}}$$

Convert from rad to deg

$\pi \text{ rad} \mapsto 180^\circ$

$\frac{180^\circ}{\pi}\,\theta \text{ rad} = (\ ?\ )^\circ$

Example: $\frac{\pi}{6} \text{ rad} \mapsto \boxed{\frac{180^\circ}{\pi}\frac{\pi}{6} = \left(\frac{180}{6}\right)^\circ = 30^\circ}$

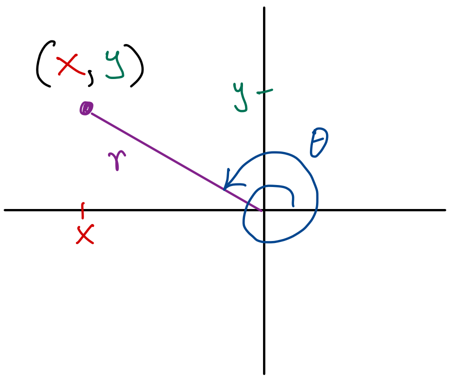
Does $\cos^{-1} x$ mean the same thing as $\sec x$?

- YES
- NO

Def of trig functions

[Go to Appendix D Trigonometry pg A 26 eq $\boxed{5}$]

<u>Def</u> When θ is any number . . .



$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{1}{\cos\theta}$$

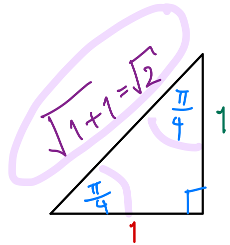
$$\tan\theta = \boxed{\frac{y}{x}} \qquad \cot\theta = \frac{1}{\tan\theta}$$

- Going counterclockwise gives positive angle
- Going clockwise gives negative angle

Computing exact values when $\theta = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$

[Go to Appendix D Trigonometry pg A26 Fig 9]

$\boxed{\theta = \frac{\pi}{4}}$

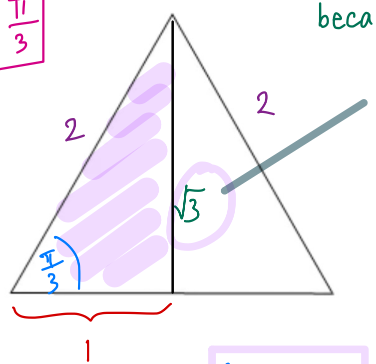


$\frac{\pi}{4}$ rad $= \left(\frac{180}{4}\right)^\circ = 45^\circ$

$\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \boxed{\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{2}$

Consider an equilateral triangle w/ sides of length 2: $2\triangle 2$, base 2

$\boxed{\theta = \frac{\pi}{3}}$



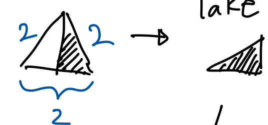
because $\sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

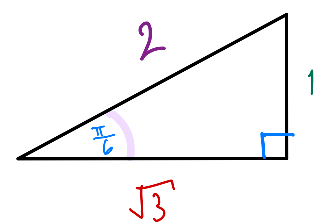
$\cos\frac{\pi}{3} = \boxed{\frac{1}{2}}$

$\frac{\pi}{3}$ rad $= \boxed{\left(\frac{180}{3}\right)^\circ = 60^\circ}$

$\boxed{\theta = \frac{\pi}{6}}$



Take half



$\sin\frac{\pi}{6} = \boxed{\frac{1}{2}}$

$\cos\frac{\pi}{6} = \boxed{\frac{\sqrt{3}}{2}}$

$\frac{\pi}{6}$ rad $= \left(\frac{180}{6}\right)^\circ = 30^\circ$

Turn sin(x) into a one-to-one function

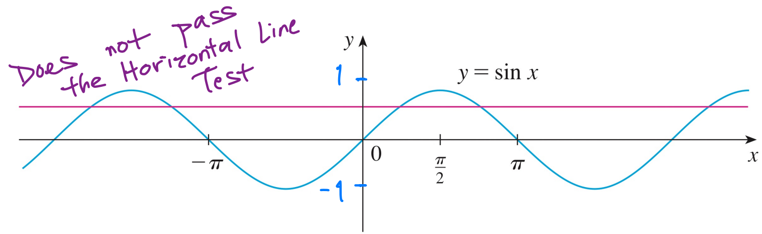


FIGURE 1 sin(x) with domain $(-\infty, \infty)$

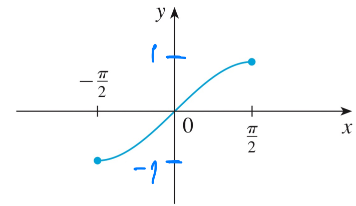


FIGURE 2
$y = \sin x, \underbrace{-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}}_{\text{Restricted domain}}$

- $g(x) = \sin x$ with $\underbrace{\text{domain}}_{\text{all possible inputs}}$ $(-\infty, \infty)$ is not one-to-one

- Define $f(x) = \sin x$ with domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then $f(x)$ is one-to-one.

$\underbrace{\text{Image/range}}_{\text{all possible outputs}}$ of f is $[-1, 1]$.

Inverse of sin(x) is called arcsin(x)

- $f(x) = \sin x$

  domain of f: $[-\frac{\pi}{2}, \frac{\pi}{2}]$

  Image of f: $[-1, 1]$

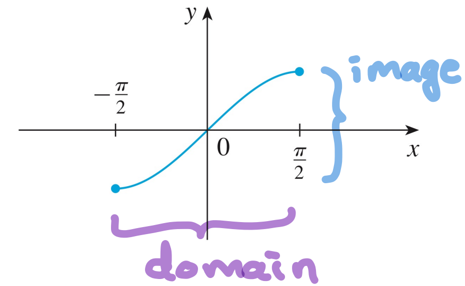


FIGURE 2
$y = \sin x,\ -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

- Let $\arcsin(x)$ or $\sin^{-1}(x)$ denote the inverse function of f ($\sin x$ with restricted domain)

  Recall (Sec 6.1) def of f^{-1} says ...

  $$f^{-1}(x) = y \iff f(y) = x$$

So, $\boxed{\arcsin(x) = y \iff \sin(y) = x \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}}$

(domain of f^{-1}) = (image of f)
(image of f^{-1}) = (domain of f)

domain of $\arcsin(x)$:	$[-1, 1]$
image of $\arcsin(x)$:	$[-\pi/2, \pi/2]$

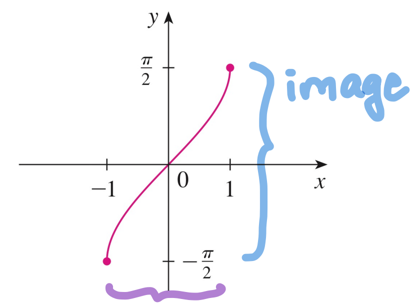


FIGURE 4
$y = \sin^{-1} x = \arcsin x$

Inverse of cos(x) is called arccos(x)

- Restrict the $\underbrace{\text{domain}}_{\text{all possible inputs}}$ of cos(x) to $[0, \pi]$.

Then this function is one-to-one.

$\underbrace{\text{Image}}_{\text{all possible outputs}}$: $[-1, 1]$

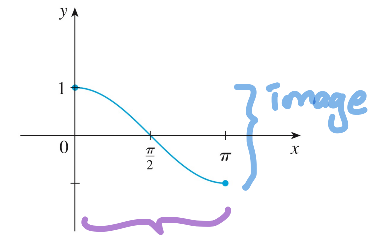


FIGURE 6
$y = \cos x,\ 0 \leqslant x \leqslant \pi$

- Let arccos(x) or $\cos^{-1}(x)$ denote the inverse function

So, $\boxed{\arccos(x) = y \iff \cos(y) = x \text{ and } 0 \leq y \leq \pi}$

domain of arccos(x): $[-1, 1]$
Image of arccos(x): $[0, \pi]$

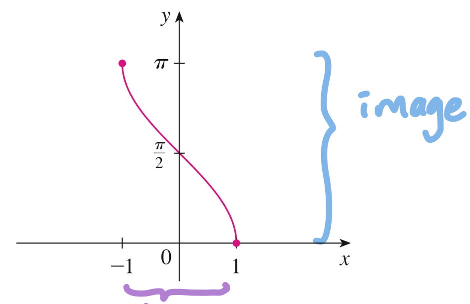


FIGURE 7
$y = \cos^{-1} x = \arccos x$

Does $\cos^{-1} x$ mean the same thing as sec(x)?

- YES
- NO

Computing exact values for arccos(x)

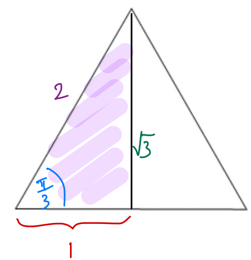
- $\arccos\left(\frac{1}{2}\right) = ?$

<u>Answer</u> (Remember that $\underbrace{\frac{1}{2}, \frac{\sqrt{3}}{2}}_{\text{for } \frac{\pi}{6}, \frac{\pi}{3}}$, $\underbrace{\frac{1}{\sqrt{2}}}_{\text{for } \frac{\pi}{4}}$ are special values)

Because
- the input is $\boxed{\frac{1}{2}}$
- know $\cos(\theta) \overset{\text{def}}{=} \boxed{\frac{x}{r}}$
- r is positive

Set $\boxed{x = 1}$ and $\boxed{r = 2}$ and draw →

$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, so $\arccos\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{3}}$



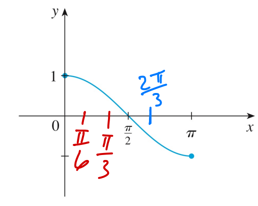


FIGURE 6
$y = \cos x, 0 \leqslant x \leqslant \pi$

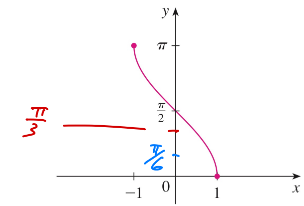


FIGURE 7
$y = \cos^{-1}x = \arccos x$

In chapter 6, 7, assume r is positive!

- $\arccos\left(\frac{-1}{2}\right) = ?$

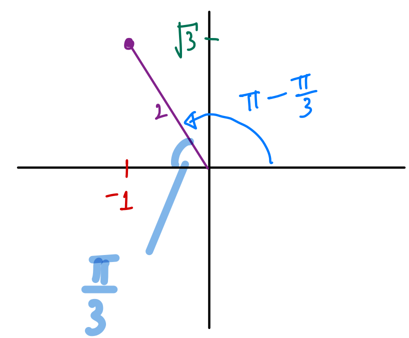
<u>Answer</u>

Because
- the input is $\boxed{-\frac{1}{2}}$
- know $\cos(\theta) \overset{\text{def}}{=} \boxed{\frac{x}{r}}$
- r is positive

Set $\boxed{x = -1}$ and $\boxed{r = 2}$ & draw →

$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$, so $\arccos\left(-\frac{1}{2}\right) = \boxed{\pi - \frac{\pi}{3} = \frac{2\pi}{3}}$

(See the next page for why $\theta = \pi - \frac{\pi}{3}$)



We know not to go to the 3rd quadrant because ...

the image of arccos(x) is $[0, \pi]$

$$\arccos\left(-\frac{1}{2}\right) = ?$$

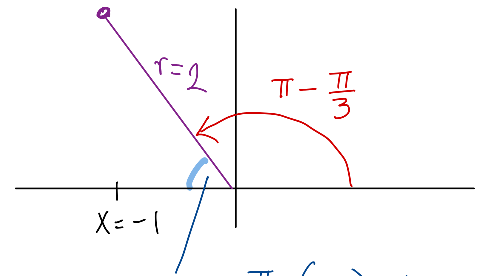
Answer

$$\cos\left(\underbrace{\boxed{\theta}}\right) = -\frac{1}{2}$$

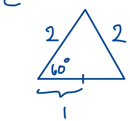
must be between 0 and π because the domain of the (restricted) $\cos(x)$ is $[0, \pi]$

$$-\frac{1}{2} \Longrightarrow \begin{cases} x = -1 \\ r = 2 \end{cases}$$

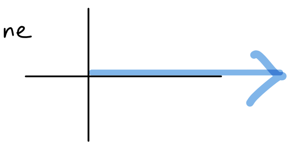
We can put the point in the 2nd or 3rd quadrant, but since θ must be in $[0, \pi]$, the point cannot be in the 3rd quadrant.



This angle is $\frac{\pi}{3}$ $(60°)$ because of the equilateral triangle



- We always measure θ from the half-line



so $\theta = \pi - \frac{\pi}{3}$

- So $\cos\left(\pi - \frac{\pi}{3}\right) = -\frac{1}{2}$,

so $\arccos\left(-\frac{1}{2}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

Practice computing exact values

- $\cos\left(\arccos\left(-\frac{1}{2}\right)\right) = \ ?$

  Answer: $-\frac{1}{2}$, since $f(f^{-1}(b)) = b$ for all b in the domain of f^{-1}

- $\arcsin\left(\sin\left(\frac{\pi}{6}\right)\right) = \ ?$

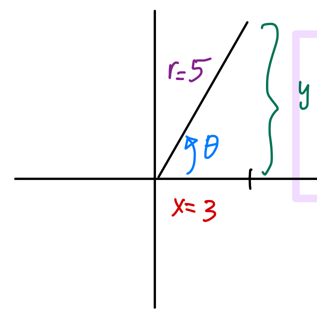
  Answer: $\frac{\pi}{6}$, since $f^{-1}(f(a)) = a$ for all a in the domain of f.

- Exercise 8 (textbook): $\csc\left(\arccos \frac{3}{5}\right) = \ ?$

  Answer: Let $\theta = \arccos \frac{3}{5}$,

  i.e. angle where $x = 3$, $r = 5$

  $$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\left(y/r\right)} = \frac{r}{y} = \frac{5}{y} = \frac{5}{4}$$



$$\boxed{\begin{aligned} y &= \sqrt{5^2 - 3^2} \\ &= \sqrt{25-9} \\ &= 4 \end{aligned}}$$

[Practice similar technique:
- Textbook Example 1
- Textbook Solution 2 of Example 2]

Sec 6.6 Inverse Trig Functions

Part a: Trigonometry Review

Part b: Calculus of arcsin (x) and arccos (x)

Part c: Calculus of other inverse trig functions

Sec 6.6 part (b) Calculus of arcsin(x) and arccos(x)

[Go to Sec 2.4 pg 148 & review]

memorize

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = (\sec x)^2$$

memorize, or compute $\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$ using product rule + chain rule

Don't memorize — More practical to leave in terms of $\sin x$ and $\cos x$ when evaluating integrals

$$\frac{d}{dx}\csc x = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = -\frac{1}{(\sin x)^2}\cos x$$

$$\frac{d}{dx}\sec x = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = -\frac{1}{(\cos x)^2}\cdot -\sin x = \frac{\sin x}{(\cos x)^2}$$

$$\frac{d}{dx}\cot x = \frac{d}{dx}\left(\frac{1}{\tan x}\right) = -\frac{1}{(\tan x)^2}\cdot(\sec x)^2$$

$$= -\left(\frac{\cos x}{\sin x}\right)^2\cdot\frac{1}{(\cos x)^2}$$

$$= -\frac{1}{(\sin x)^2}$$

$$= -(\csc x)^2$$

Compute $\frac{d}{dx}\left[\arccos(x)\right]$

(Use implicit differentiation)

Let $y = \arccos(x)$

$\cos(y) = x$

$\frac{d}{dx} \cos y = \frac{d}{dx} x$

$-\sin y \cdot \frac{dy}{dx} = 1$

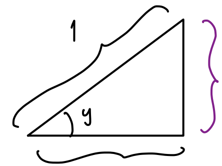
$\frac{dy}{dx} = -\frac{1}{\sin y}$

$= -\frac{1}{\sqrt{1-x^2}}$

(because $\sin y = \sqrt{1^2 - x^2}$)

$\Longrightarrow$

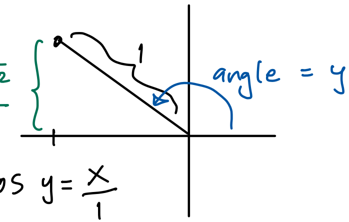
if x is in $(0, 1)$



$\frac{\sqrt{1^2-x^2}}{1} = \sin y$

$\cos y = \frac{x}{1}$

if x is in $[-1, 0]$



$\sin y = \frac{\sqrt{1^2-x^2}}{1}$

$\cos y = \frac{x}{1}$

$$\boxed{\frac{d}{dx}\left[\arccos(x)\right] = -\frac{1}{\sqrt{1-x^2}}}$$

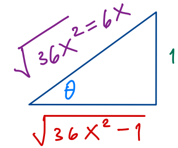
[<u>Practice similar technique</u>:

See textbook eq $\boxed{3}$ or recommended Khan Academy for $\frac{d}{dx}\left[\sin^{-1}(x)\right]$]

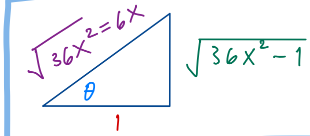
Webwork Prob 5

Evaluate $\displaystyle\int \frac{1}{x\sqrt{36x^2-1}}\,dx$

Step 1: Draw $\rightarrow$



OR



I choose

either picture will give a correct result

Step 2 Convert all to θ

$$\frac{1}{6x}=\frac{adj}{hyp}=\cos\theta \Rightarrow x=\frac{1}{6}\,\frac{1}{\cos\theta}$$

$$\Downarrow$$

$$\boxed{\frac{1}{x}=6\cos\theta}$$

$$\Downarrow$$

$$dx=\frac{1}{6}\left(-\frac{1}{(\cos\theta)^2}\right)\frac{d}{d\theta}\cos\theta\,d\theta$$

$$=-\frac{1}{6}\,\frac{1}{(\cos\theta)^2}\left(-\sin\theta\right)d\theta$$

$$dx=\frac{1}{6}\,\frac{\sin\theta}{(\cos\theta)^2}\,d\theta$$

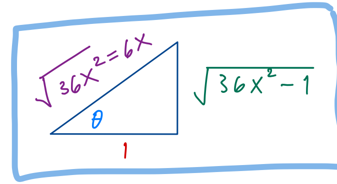
$$\frac{\sqrt{36x^2-1}}{1}=\frac{opp}{adj}=\tan\theta=\frac{\sin\theta}{\cos\theta}\Rightarrow \boxed{\frac{1}{\sqrt{36x^2-1}}=\frac{\cos\theta}{\sin\theta}}$$

$$\int \boxed{\frac{1}{x}}\,\boxed{\frac{1}{\sqrt{36x^2-1}}}\,dx=$$

Webwork Prob 5

Evaluate $\displaystyle\int \frac{1}{x\sqrt{36x^2-1}}\,dx$

Step 1. Draw $\rightarrow$



Step 2 Convert all to θ

$$\frac{1}{6x} = \frac{\text{adj}}{\text{hyp}} = \cos\theta \Rightarrow x = \frac{1}{6}\,\frac{1}{\cos\theta} \iff \cos\theta = \frac{1}{6x}$$

$$\boxed{\frac{1}{x} = 6\cos\theta}$$

$$dx = \frac{1}{6}\,\frac{\sin\theta}{(\cos\theta)^2}\,d\theta$$

$$\boxed{\frac{1}{\sqrt{36x^2-1}} = \frac{\cos\theta}{\sin\theta}}$$

$$\int \boxed{\frac{1}{x}}\,\boxed{\frac{1}{\sqrt{36x^2-1}}}\,dx = \int (6\cos\theta)\left(\frac{\cos\theta}{\sin\theta}\right)\frac{1}{6}\,\frac{\sin\theta}{(\cos\theta)^2}\,d\theta$$

$$= \int d\theta$$

$$= \theta + C$$

$$= \arccos\left(\frac{1}{6x}\right) + C$$

$\theta = \arccos\left(\frac{1}{6x}\right)$ because $\frac{1}{6x} = \cos\theta$

Check: $\displaystyle\frac{d}{dx}\arccos\left(\frac{1}{6x}\right) = -\frac{1}{\sqrt{1-\left(\frac{1}{6x}\right)^2}}\cdot\frac{1}{6}\left(-\frac{1}{x^2}\right)$

$= \vdots$

$\stackrel{?}{=} \dfrac{1}{x}\,\dfrac{1}{\sqrt{36x^2-1}}$

Sec 6.6 Inverse Trig Functions

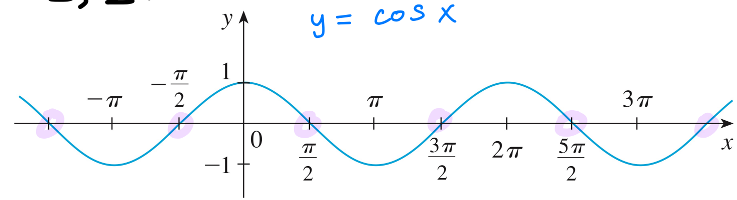
Part a: Trigonometry Review

Part b: Calculus of arcsin(x) and arccos(x)

$\boxed{\text{Part c: Calculus of other inverse trig functions}}$

Sec 6.6 part (c) Calculus of other inverse trig functions

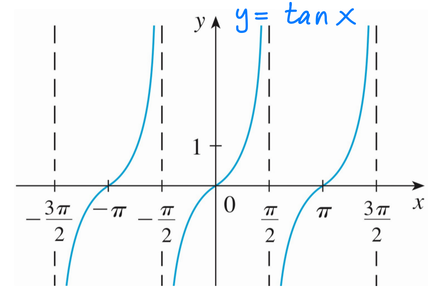
Restrict domain of tan(x) to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



- $\cos(x) = 0$ if and only if

$\boxed{x = \frac{k\pi}{2}}$ for some odd integer k

- $\tan x = \boxed{\dfrac{\sin x}{\cos x}}$



has domain all numbers x such that $\cos x \neq 0$,

i.e. all numbers except for $k\frac{\pi}{2}$ where k is an odd integer.

the most natural choice

- Define $f(x) = \tan x$ with domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Then $f(x)$ is one-to-one.

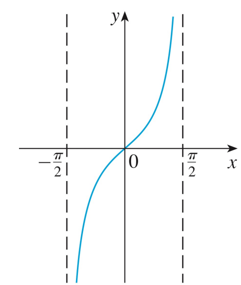


FIGURE 8
$y = \tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$

arctan(x) definition

- Denote the inverse of tan(x)

by $\boxed{\arctan(x) \text{ or } \tan^{-1}(x)}$

Warning: $\tan^{-1}(x)$ is <u>NOT</u> $[\tan(x)]^{-1}$

- $(\tan(x))^{-1} = \dfrac{1}{\tan(x)}$
- $\tan^{-1}(x)$ means the inverse of tan(x)

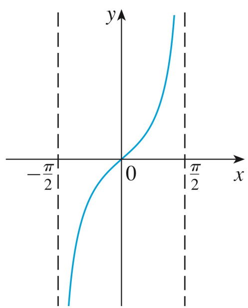
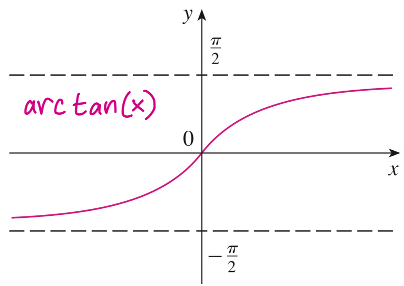


FIGURE 8
$y = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$

Reflect across y = x line



(restricted) tan x

Domain: $\boxed{\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$

Image/range: $\boxed{\text{all numbers}}$

arctan x

Domain: $\boxed{\text{all numbers}}$

Image/range: $\boxed{\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$

Is arctan x the same as $\dfrac{1}{\tan x}$?

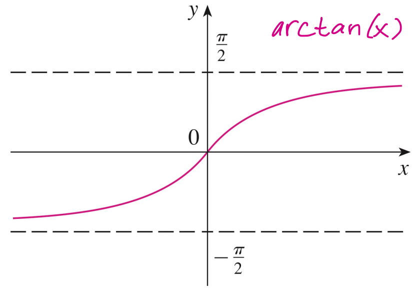
- YES
- NO

Compute $\frac{d}{dx}[\arctan(x)]$

Step 1

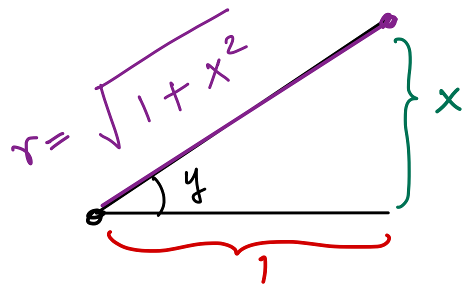
Let $y = \arctan(x)$

$\tan(y) = x$



Step 2

"Draw a triangle"



(con't below for Steps 3, 4)

Compute $\frac{d}{dx}\left[\arctan(x)\right]$

Step 1

Let $y = \arctan(x)$

$\tan(y) = x$

Step 3

Implicit Differentiation

$$\frac{d}{dx}\left(\tan y\right) = \frac{d}{dx}(x)$$

Recall

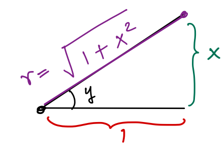
$$\frac{d}{d\theta}\tan\theta = (\sec\theta)^2$$

$$(\sec y)^2 \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{(\sec y)^2}$$

$$\stackrel{\Delta}{=} (\cos y)^2$$

Step 4 Express $\frac{dy}{dx}$ In terms of x



$$\cos y = \boxed{\frac{\text{adj}}{r} = \frac{1}{\sqrt{1+x^2}}}$$

$$(\cos y)^2 \underset{(*)}{=} \boxed{\frac{1}{1+x^2}}$$

$$\frac{dy}{dx} \stackrel{\Delta}{=} (\cos y)^2$$

$$\underset{(*)}{=} \boxed{\frac{1}{1+x^2}}$$

$$\boxed{\frac{d}{dx}\left[\arctan(x)\right] = \frac{1}{1+x^2}}$$

Webwork Prob 13 (my version)

Evaluate $\displaystyle\int \frac{-8 \sin x}{1+(\cos x)^2}\,dx$

Try u- substitution

Try • $u = \sin x$ $\quad du = \cos x \, dx$

may work $\boxed{\bullet\ u = \cos x \quad du = -\sin x \, dx}$

• $u = (\cos x)^2$ $\quad du = 2 \cdot \cos x \,(-\sin x)\, dx$

• $u = \dfrac{1}{1+(\cos x)^2}$ $\quad du = -\dfrac{1}{[1+(\cos x)^2]^2}\, 2\cos x \cdot (-\sin x)\, dx$

$$\left[\begin{array}{l} \dfrac{1}{1+(\cos x)^2} = \dfrac{1}{1+u^2} \\ -8 \sin x \, dx = 8\, du \end{array}\right]$$

$$\int \frac{-8 \sin x}{1+(\cos x)^2}\,dx = \int \frac{1}{1+u^2} \cdot 8 \, du$$

make sure there are no x (only u)

$$\left[\begin{array}{l} \text{Since } \dfrac{d}{du} \arctan u = \dfrac{1}{1+u^2} \\ \text{we know} \\ \displaystyle\int \frac{1}{1+u^2}\,du = \arctan u + C \end{array}\right] = 8 \arctan u + C$$

$$= \boxed{8 \arctan(\cos x) + C}$$

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Problems requiring similar technique:

- Textbook Example 8, 9
- Webwork Prob 3, 7, 12