Sec 6.4 (a) Derivatives of logarithmic functions

knowing that $\frac{d}{dx} e^x = \boxed{e^x}$,

use Implicit Differentiation to compute $\frac{d}{dx} \ln x$

Let $y = \ln x$

Apply $e^{\square}$ to both sides:

$$e^y = e^{\ln x}$$

$$e^y \overset{*}{=} \boxed{x}$$ (since e^x and $\ln x$ are $\boxed{\text{inverse}}$ functions)

Differentiate with respect to x:

$$\frac{d}{dx}(e^y) = \frac{d}{dx}(x)$$

chain rule

$$e^y \boxed{\frac{dy}{dx}} = \boxed{1}$$

$$\frac{dy}{dx} = \boxed{\frac{1}{e^y} = \frac{1}{x}}$$

by *

Memorize!

$$\boxed{\frac{d}{dx} \ln x = \boxed{\frac{1}{x}}}$$

Textbook eq $\boxed{1}$

knowing that $\frac{d}{dx} \ln x \overset{\text{eq } \boxed{1}}{=} \boxed{\frac{1}{x}}$, compute $\frac{d}{dx}\left[\ln\left(\begin{matrix}\text{some}\\ \text{function}\\ \text{of } x\end{matrix}\right)\right]$

<u>Ex 1</u> Evaluate $\frac{d}{dx} \ln \sqrt[5]{x}$

$$\frac{d}{dx} \ln \sqrt[5]{x} = \frac{d}{dx}\left(\ln\left(x^{\frac{1}{5}}\right)\right)$$

$$= \frac{d}{dx}\left(\frac{1}{5} \ln (x)\right)$$

$$= \boxed{\frac{1}{5} \frac{1}{x}}$$

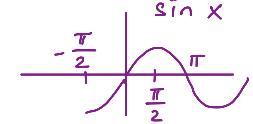
Knowing that $\frac{d}{dx} \ln x = \boxed{\frac{1}{x}}$ (eq $\boxed{1}$), compute $\frac{d}{dx}\left[\ln\left(\text{some function of } x\right)\right]$

Review: $\sin x$, $\cos x$, $-\sin x$, $-\cos x$, $\sin x$, $\vdots$

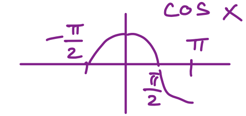
$\frac{d}{dx} \sin x = \boxed{\cos x}$

$\frac{d}{dx} \cos x = \boxed{\sin x}$

If you forget where the minus signs should go, sketch the graphs



increasing, so derivative is pos



positive values

Review: $\tan x = \boxed{\frac{\sin x}{\cos x}}$, $\cot x = \boxed{\frac{\cos x}{\sin x}}$

Ex 2

Find $\frac{d}{dx} \ln(\sin x)$

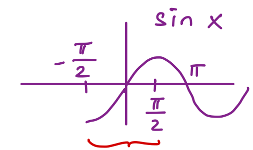
Take a minute to attempt this on your own.

Use eq $\boxed{1}$ and Chain rule.

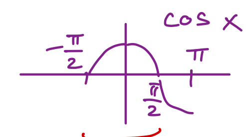
Knowing that $\frac{d}{dx}\ln x \overset{\text{eq}\boxed{1}}{=} \frac{1}{x}$, compute $\frac{d}{dx}\left[\ln\left(\begin{matrix}\text{some}\\ \text{function}\\ \text{of } x\end{matrix}\right)\right]$

Review: $\sin x$, $\frac{d}{dx}\sin x = \cos x$
$\cos x$, $\frac{d}{dx}\cos x = -\sin x$
$-\sin x$
$-\cos x$
$\sin x$
$\vdots$

If you forget where the minus signs should go, sketch the graphs



increasing, so derivative is pos



positive values

Review: $\tan x = \frac{\sin x}{\cos x}$, $\cot x = \frac{\cos x}{\sin x}$

Ex 2 Find $\frac{d}{dx}\ln(\sin x)$

Answer

$$\frac{d}{dx}\ln(\sin x) = \frac{1}{\boxed{\sin x}} \cdot \boxed{\frac{d}{dx}(\sin x)}$$

by eq $\boxed{1}$ and Chain Rule

$$= \boxed{\frac{1}{\sin x} \cdot \cos x}$$

(It's ok to stop here)

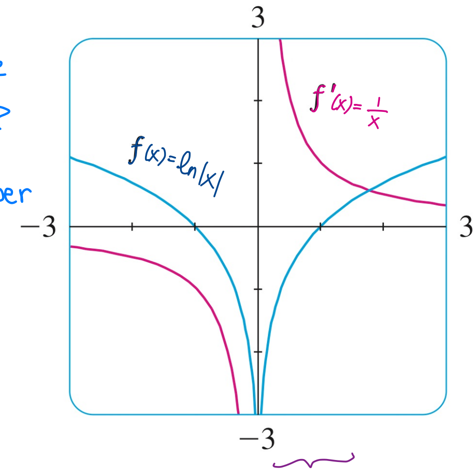
$$= \boxed{\cot x}$$

Ex 3

Let $f(x) := \ln|x|$ with

domain $\underbrace{(-\infty, 0) \cup (0, \infty)}_{\text{all numbers except } 0}$

Picture to help you remember



This means

$$f(x) = \begin{cases} \boxed{\ln x} & \text{if } x > 0 \\ \boxed{\ln(-x)} & \text{if } x < 1 \end{cases}$$

So

$$f'(x) = \begin{cases} \boxed{\frac{1}{x}} & \text{if } x > 0 \\ \boxed{\frac{1}{-x}(-1) = \frac{1}{x}} & \text{if } x < 1 \end{cases}$$

So $f'(x) = \boxed{\frac{1}{x}}$ for all $x \neq 0$

Textbook Eq $\boxed{4}$

$$\boxed{\int \frac{1}{x}\, dx = \ln|x| + C}$$

When x is small & pos, $y = \ln|x|$ is very steep with pos slope, so $f'(x)$ is big and positive

When $|x|$ is small & x is neg, $y = \ln|x|$ is very steep with neg slope, so $|f'(x)|$ is big & $f'(x)$ is neg.

Memorize!

Textbook Eq $\boxed{4}$

$$\boxed{\int \frac{1}{x}\,dx = \ln|x| + C}$$

Recall: $\cot x = \dfrac{\cos x}{\sin x}$

$\sin x$
$\cos x$
$-\sin x$
$-\cos x$
$\sin x$
$\vdots$

$\frac{d}{dx}\sin x = \cos x$

$\frac{d}{dx}\cos x = -\sin x$

Ex 4

Calculate $\int \cot x \, dx$

Answer

$$\int \cot x \, dx = \int \boxed{\frac{\cos x}{\sin x}}\,dx$$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + C$$

$$= \ln|\sin x| + C$$

Try u-substitution

Try $\cancel{u = \cos x}$ $\quad du = -\sin x \, dx$

Try $u = \sin x \quad du = \cos x \, dx$

looks good

* Practice a similar problem: Textbook Example 11, this Sec 6.4

Practice using $\int \frac{1}{x} = \ln|x| + C$

(Webwork Prob 8) Evaluate $\displaystyle\int \frac{1}{x(\ln x)^2}\,dx$

Hint: Try u-sub

(works) Try $u = \ln x$ $\quad du = \frac{1}{x}\,dx$

No Try $u = \frac{1}{x}$ $\quad du = -\frac{1}{x^2}\,dx$

No Try $u = \left(\frac{1}{\ln x}\right)$ $\quad du = -\frac{1}{(\ln x)^2} \cdot \frac{1}{x}\,dx$

$$\int \frac{1}{u^2}\,du = \int u^{-2}\,du$$

$$= \frac{u^{-2+1}}{-1} + C$$

$$= -u^{-1} + C$$

$$\int \frac{1}{x(\ln x)^2}\,dx = -\frac{1}{\ln x} + C$$

6.4(b) Logarithmic Differentiation (new technique)

(Webwork Prob 6) Q: $f(x) = (\sin(x))^x$. Find $f'(1)$.

Apply Logarithmic differentiation

1.) $y = (\sin(x))^x$

Take natural logarithms of both sides:

$$\ln[y] = \ln\left[(\sin x)^x\right]$$

Simplify right hand side using properties of $\ln$:

$$\ln[y] = x \ln(\sin(x)), \quad \text{since} \quad \ln(\square^r) = r\ln(\square)$$

2) Differentiate implicitly with respect to x:

$$\underbrace{\frac{d}{dy}(\ln[y])}\frac{dy}{dx} = \underbrace{\frac{d}{dx}\left[x \ln(\sin x)\right]}$$

$$\frac{1}{y}\frac{dy}{dx} = x \cdot \underbrace{\frac{d}{dx}(\ln(\sin x))} + \underbrace{\frac{d}{dx}(x)} \cdot \ln(\sin x)$$

$$= x \cdot \frac{1}{\sin x} \cdot \underbrace{\frac{d}{dx}(\sin x)} + 1 \cdot \ln(\sin x)$$

$$= x \frac{1}{\sin x} \cos x + \ln(\sin x)$$

3) Solve for $\frac{dy}{dx}$:

Multiply by y: $\frac{dy}{dx} = y\left(x \cdot \frac{1}{\sin x} \cos x + \ln(\sin x)\right)$

$$= (\sin x)^x\left(x \cdot \frac{1}{\sin x}\cos x + \ln(\sin x)\right)$$

So $\boxed{f'(1) = (\sin 1)\left(\frac{\cos 1}{\sin 1} + \ln(\sin 1)\right)}$

Recall Sec 6.2: If $g(x)=b^x$ then $g'(x)=g'(0)\cdot b^x$

If $b=e$ then $g'(0)=1$ so $\frac{d}{dx}(e^x)=e^x$

Q: If $b\neq e$, what is $\frac{d}{dx}(b^x)$? Note $\frac{d}{dx}(b^x)$ is NOT b^x if $b\neq e$.

Ex what is $\frac{d}{dx}(5^x)$ or $\frac{d}{dx}\left(\frac{6}{7}\right)^x$? $\frac{d}{dx}(5^x)$ is not 5^x since $5\neq e$

Answer Let $y=b^x$ where $b\neq 1$.

$y=e^{[\ln(b)]x}$ by Fact ⑥ in previous page

Note: $\ln(b)$ is a constant,

so we can treat it like any number

$$\frac{dy}{dx}=\ln(b)\cdot e^{[\ln(b)]x} \quad \left(\text{chain Rule: } \frac{d}{dx}(e^{cx})=ce^{cx}.\right)$$

Fact ⑥ says $b^x=e^{[\ln(b)]x}$

so $\frac{dy}{dx}=\ln(b)\cdot b^x$,

so $$\boxed{\frac{d}{dx}(b^x)=\ln(b)\cdot b^x}$$

(Textbook reference: eq $\boxed{7}$)

Ex a. $\frac{d}{dx}5^x=\ln 5\cdot 5^x$

b. $$\frac{d}{dx}\left((e^2)^x\right)=\ln(e^2)\cdot(e^2)^x=2\underbrace{\ln(e)}_{1}\cdot e^{2x}=2e^{2x}.$$

From $\boxed{\frac{d}{dx}(b^x) = \ln(b).\, b^x}$, previous page

we get ... $b^x + C = \int \ln(b)\, b^x\, dx$

$$\frac{1}{\ln(b)}\left(b^x + C\right) = \int b^x\, dx$$

$$\boxed{\frac{b^x}{\ln b} + D = \int b^x\, dx}$$

Ex
$$\int_0^5 2^x\, dx = \frac{2^x}{\ln 2}\Bigg]_{x=0}^{x=5}$$

$$= \frac{2^5}{\ln 2} - \frac{2^0}{\ln 2}$$

$$= \frac{2^5 - 1}{\ln 2}$$

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use Implicit Differentiation to compute $\frac{d}{dx}(\log_b x)$

Let $y = \log_b x$, then apply $b^{\square}$ to both sides:

$$b^y = b^{(\log_b x)}$$

$$b^y \overset{**}{=} \boxed{x} \quad \text{(because } b^x \text{ and } \log_b x \text{ are inverse functions)}$$

Differentiate with respect to x:

$$\frac{d}{dx}\left(b^y\right) = \frac{d}{dx}\left(x\right)$$

$$\Big(\underbrace{b^y(\ln b)}_{\text{by } (*)}\Big) \cdot \Big(\underbrace{\frac{dy}{dx}}_{\text{because of chain rule}}\Big) = \boxed{1}$$

$$\left(\underbrace{x}(\ln b)\right) \cdot \left(\frac{dy}{dx}\right) = \boxed{1} \quad \left(\text{since } b^y \overset{**}{=} x\right)$$

$$\frac{dy}{dx} = \boxed{\frac{1}{x(\ln b)}} \quad \text{so} \quad \boxed{\frac{d}{dx}(\log_b x) = \frac{1}{x(\ln b)}}$$

More practice, similar to Webwork

Differentiate $y = (4+2x^2)^{\ln(x)}$

Answer $\ln y = \ln\left((4+2x^2)^{\ln(x)}\right)$

$\ln y = \boxed{\ln(x)\ \ln(4+2x^2)}$ (because $\ln(\square^r) = r\ln(\square)$)

Apply Implicit Differentiation

$$\frac{1}{y}\frac{dy}{dx} = \left(\frac{d}{dx}\ln(x)\right)\ln(4+2x^2) + \ln(x)\cdot\frac{d}{dx}\left(\ln(4+2x^2)\right)$$

by eq $\boxed{1}$ $\frac{d}{dy}\ln y = \frac{1}{y}$

$$\frac{1}{y}\frac{dy}{dx} = \boxed{\frac{1}{x}}\ \ln(4+2x^2) + \ln(x)\cdot\boxed{\frac{1}{4+2x^2}\cdot 4x}$$

Multiply by y: $$\frac{dy}{dx} = \boxed{y\cdot\left(\frac{\ln(4+2x^2)}{x} + \frac{4x\ \ln(x)}{4+2x^2}\right)}$$

$$\frac{dy}{dx} = \boxed{(4+2x^2)^{\ln(x)}\left(\frac{\ln(4+2x^2)}{x} + \frac{4x\ \ln(x)}{4+2x^2}\right)}$$

Practice applying Logarithmic Differentiation (new technique)

Differentiate $y = x^{\sqrt{x}}$ Hint: Start by applying ln to both sides then apply implicit differentiation)

(Read solution in Textbook Sec 6.4 Example 16)