Sec 6.3 Logarithmic functions

Goal : inverse of the natural exponential function e^x
& its properties

Q: what do you call trees that have beat ?
A:

Def The inverse function of e^x is called the <u>natural logarithm</u>, denoted by ln.

(Recall $f^{-1}(x) = y \iff f(y) = x$)

$$\underbrace{\ln(x)}_{} = y \iff e^y = x$$

Read "natural log of x"

Def In general, if b is positive and $b \neq 1$, then b^x is 1-1.
The inverse function of b^x is denoted $\log_b x$.

$$\underbrace{\log_b(x)}_{} = y \iff b^y = x.$$

Read "log base b of x"

Note: $\ln(x)$ is shorthand for $\log_e x$

- Think: $\log_{(\text{base})}(x) = (\text{exponent}) \iff (\text{base})^{(\text{exponent})} = x$

* Domain of b^x = all real numbers = image of $\log_b x$

* Image of b^x = all positive numbers = domain of $\log_b x$
(if $b \neq 1$)

Properties of logarithmic functions

Recall: f^{-1} compose f is the identity function.
(Sec 6.1) f compose f^{-1} is also the identity function.

1) $\ln(e^x) = x$ for all x in {all real numbers} (domain of e^x)
$\log_b(b^x) = x$

2) $e^{\ln(x)} = x$ for all x in {all positive numbers} (domain of $\ln x$)
$b^{(\log_b(x))} = x$

3) Let $\ln(x) = M$, $\ln(x^r) = A$
so $x = e^M$, $x^r = e^A$

So $(e^M)^r = x^r = e^A$

Recall $(e^M)^r = e^{Mr}$ (Sec 6.2)

Memorize!

So $e^{Mr} = e^A$

Apply $\ln$ to both sides:

$rM = A$

So $\boxed{r \ln(x) = \ln(x^r)}$

e.g. $2\ln 5 = \ln(5^2)$

4) Let $\ln(x) = M$, $\ln(y) = N$
Then $x = e^M$, $y = e^N$

$xy = e^M e^N = e^{M+N}$ (Sec 6.2) properties of exponential functions

$xy = e^{M+N}$

$\ln(xy) = \ln(e^{M+N}) = M+N$

So $\boxed{\ln(xy) = \ln(x) + \ln(y)}$

e.g. $\ln(15) = \ln(3) + \ln(5)$

5) Write $\log_b x$ in terms of $\ln(\square)$

Let $y = \log_b x$
Then $b^y = x$

Apply $\ln(\square)$ to both sides:

$\ln(b^y) = \ln(x)$

$y \ln(b) = \ln(x)$ by fact 3

So $y = \dfrac{\ln(x)}{\ln(b)}$

$$\boxed{\log_b(x) = \frac{\ln(x)}{\ln(b)}}$$

e.g. $\log_{10} 5 = \dfrac{\ln(5)}{\ln(10)}$

6) Write b^x in terms of $e^{\square}$ & $\ln(\square)$:

Let $y = b^x$

Apply $\ln(\square)$ to both sides:

$\ln(y) = \ln(b^x) = x\ln(b)$ (By Fact 3)

Apply $e^{(\square)}$ to both sides:

$e^{\ln y} = e^{x\ln(b)}$

So $y = e^{x\ln(b)}$

e.g. $\left(\frac{2}{3}\right)^x = e^{x\ln\left(\frac{2}{3}\right)}$

$\boxed{b^x = e^{(x\ln(b))}} = e^{(\ln(b)\,x)}$

To get the graph of f^{-1}, reflect the graph of f about the line $y = x$

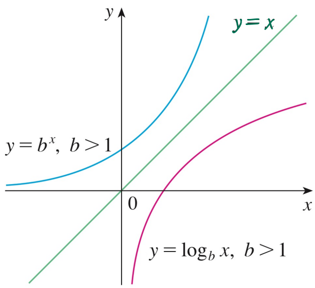

FIGURE 1

The graph of $y = \log_b x$ is the reflection of the graph of $y = b^x$ about the line $y = x$

Identity function

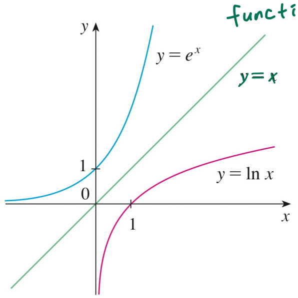

FIGURE 3

The graph of $y = \ln x$ is the reflection of the graph of $y = e^x$ about the line $y = x$.

Let $f(x) = \log_5 x$.

- The domain of $f(x)$ is $\boxed{}$

Answer: the interval $(0, \infty)$ or all positive numbers

- The image of $f(x)$ is $\boxed{}$

Answer: the interval $(-\infty, \infty)$ or $\mathbb{R}$ or all real numbers

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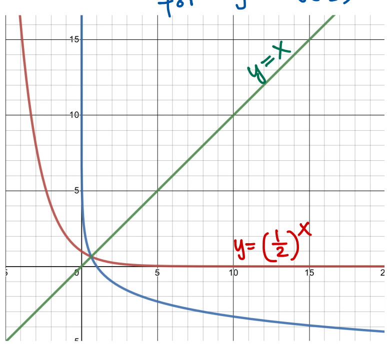
Vertical asymptote for $y = \log_{(\frac{1}{2})}(x)$

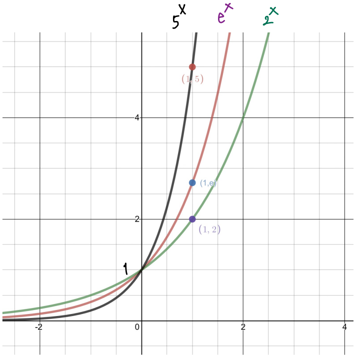


horizontal asymptote for $y = \left(\frac{1}{2}\right)^x$

$y = \log_{(\frac{1}{2})}(x)$

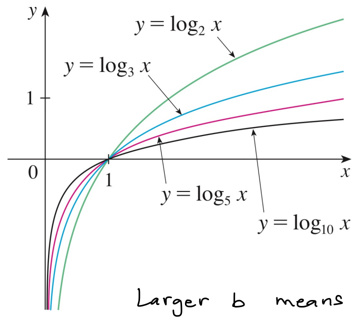
Graph of $y = \log_b(x)$ is the reflection of the graph of $y = b^x$ about the line $y = x$

Exponential function b^x



Larger b means $\boxed{\text{faster}}$ growth

Logarithmic function $\log_b x$



Larger b means $\boxed{\text{slower}}$ growth

Thm

Let $b > 1$. Then

- $\lim_{x\to\infty} \log_b x = \boxed{\infty}$
- $\lim_{x\to 0^+} \log_b x = \boxed{-\infty}$

These mean the function is not bounded!

Let $b > 1$. Then $\log_b x \to \infty$ as $x \to \infty$, but this happens very slowly.

In fact, let's compare $\sqrt{x}$ and $\ln x = \log_e x$

At first the two graphs $y = \sqrt{x}$ and $y = \ln x$

grow at comparable rates,

but eventually $\sqrt{x}$ surpasses $\ln x$

x	1	2	5	10	50	100	500	1000	10,000	100,000
$\ln x$	0	0.69	1.61	2.30	3.91	4.6	6.2	6.9	9.2	11.5
$\sqrt{x}$	1	1.41	2.24	3.16	7.07	10.0	22.4	31.6	100	316
$\frac{\ln x}{\sqrt{x}}$	0	0.49	0.72	0.73	0.55	0.46	0.28	0.22	0.09	0.04

$$\frac{\ln x}{x^{\frac{1}{2}}} \to \boxed{} \quad \text{as } x \to \infty$$

(We'll compute this in Sec 6.8)

(The limit turns out to be 0)

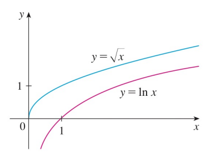


FIGURE 5

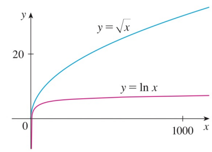


FIGURE 6

—— the end of Sec 6.3 ——

Sec 6.3 Recommended reading:

Textbook Example 1, 2, 4, 5, 6.