Sec 6.2 Exponential functions + their derivatives

The natural exponential function e^x is the most important (and "natural") function in Calculus

Coming up next: The inverse of exponential functions (the logarithmic function)

Def An exponential function is of the form $f(x) = b^x$

$$f(\text{exponent}) = \underbrace{\text{base}}_{\text{a positive number}}^{\overbrace{\text{exponent}}^{\text{variable}}}$$

Ex

- $f(x) = 2^x$, $f(x) = \left(\frac{1}{2}\right)^x$ are exponential functions
- $f(x) = x^2$, $f(x) = x^{\frac{1}{2}} = \sqrt{x}$ are not exponential functions; They are power functions.

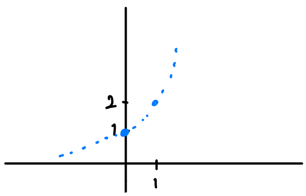
Def What does b^x mean?

- If $x = 0$, $b^x = b^0 = 1$
- If $x = 4$, $b^x = b^4 = \underbrace{b\,b\,b\,b}_{4 \text{ times}}$
- If $x = n$ is a positive integer, $b^x = b^n = \underbrace{b\,b \ldots b}_{n \text{ times}}$
- If $x = -n$ is a negative integer, $b^x = b^{-n} = \dfrac{1}{b^n} = \dfrac{1}{\underbrace{b.b\ldots b}_{n \text{ times}}}$
- If $x = \frac{2}{5}$, $b^x = b^{\frac{2}{5}} = \sqrt[5]{b^2} = \left(\sqrt[5]{b}\right)^2$
- If $x = \frac{p}{q}$ is a $\underbrace{\text{rational number}}_{\text{meaning a fraction}}$, $b^x = b^{\frac{p}{q}} = \sqrt[q]{b^p} = \left(\sqrt{b^p}\right)^q$

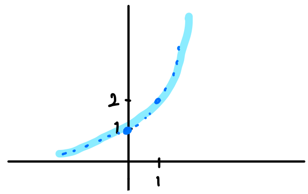
- If x is irrational (meaning, x cannot be written as a fraction), like $x = \sqrt{3}$, $x = \pi$?

Idea: Plot the points for the rational numbers only, then "fill in" the holes.

Graph of $y = 2^x$ where x is rational



"Fill in" the holes



<u>Def</u> If b is a positive number, define $b^x = \lim_{r \to x} b^r$ where the limit is over the rational numbers

Why does this Def makes sense?

Idea: Any irrational number can be approximated as closely as we like by a rational number

<u>Example</u> π is the limit of the sequence of rational numbers

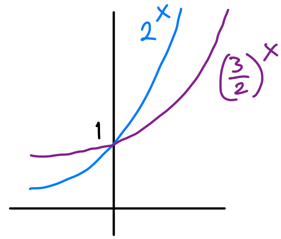
$$3.14 < 3.141 < 3.1415 < 3.1415926 < \cdots$$

"We can find a rational number which is as close to π as we like"

Sketching graphs of b^x

1) If $b > 1$, $f(x) = b^x$ is always increasing.

- Larger base b means b^x grows more rapidly for $x > 0$.



- $2 > \frac{3}{2}$ so 2^x grows more rapidly for > 0
- $\lim\limits_{x \to \infty} b^x = \infty$, $\lim\limits_{x \to -\infty} b^x = 0$

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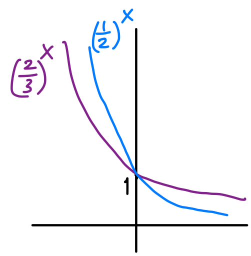
2) If $b < 1$, $f(x) = b^x$ is always decreasing.

- Larger base b means b^x is bigger for $x > 0$



- $\frac{2}{3} > \frac{1}{2}$ so $\left(\frac{2}{3}\right)^x$ is bigger than $\left(\frac{1}{2}\right)^x$ for $x > 0$.
- $\lim\limits_{x \to \infty} b^x = 0$, $\lim\limits_{x \to -\infty} b^x = \infty$.

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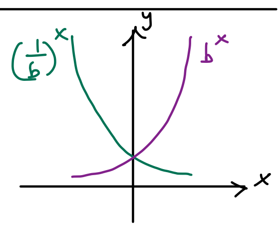
3) Graph of $y = b^{-x}$ is the reflection of the graph of $y = b^x$ about the y-axis.



Why? (x, b^{-x}) is a point in the graph of $y = b^{-x}$

$(-x, b^{-x})$ is a point in the graph of $y = b^x$

$\boxed{\text{Remember}}$

If $b > 1$, the exponential function b^x grows far more rapidly than power functions!

Thm If b is a positive number and $b \neq 1$, then

* $f(x) = b^x$ is continuous
* domain of f is all real numbers $(-\infty, \infty)$

  all possible inputs of f
* image of f is $(0, \infty)$

  all possible outputs of f
* $b^{x+y} = b^x b^y$
* $(b^x)^y = b^{xy}$
* $(ab)^x = a^x b^x$ (if a is also a positive number)

Thm If b is positive & $f(x) = b^x$,

then $f'(x) = (\text{constant})\, b^x$.

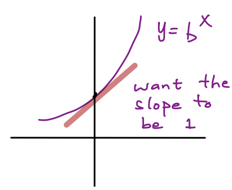
"rate of change of b^x is proportional to b^x"

In fact, $\boxed{f'(x) = f'(0)\, b^x}$

The bigger b is, the bigger this $f'(0)$ is

Most exciting definition

If we consider all positive numbers b, there is exactly one number b such that the slope of $y = b^x$ at $x = 0$ is equal to 1.



This number is called e. It's close to 2.718.
e is transcendental, meaning we can't use just algebra to define it.

We say $\boxed{f(x) = e^x \text{ is the natural exponential function}}$

From above thm, we have $f'(x) = f'(0)\, e^x = 1 \cdot e^x$, so $\boxed{\dfrac{d}{dy} e^x = e^x}$

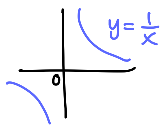
Fact The <u>only</u> function which is equal to its own derivative is $f(x) = \underbrace{A}_{\text{some constant.}} e^x$

Ex 1 Let $f(x) = e^{\frac{1}{x}}$. Domain of f is all nonzero real numbers. Find asymptotes, $f'(x)$, $f''(x)$, then sketch its graph.

$f(x)$ is not defined at $x=0$, so check if there is a vertical asymptote as $x \to 0^+$ or $x \to 0^-$.

* $\lim\limits_{x \to 0^+} e^{\frac{1}{x}} = ?$

Recall:



$\boxed{\text{As } x \to 0^+, \ \frac{1}{x} \to \infty \quad \text{call this } t}$

so $\lim\limits_{x \to 0^+} e^{\frac{1}{x}} = \lim\limits_{t \to \infty} e^t = \infty$

So $x=0$ is a vertical asymptote from the right

* $\lim\limits_{x \to 0^-} e^{\frac{1}{x}} = ?$

$\boxed{\text{As } x \to 0^-, \ \frac{1}{x} \to -\infty}$

so $\lim\limits_{x \to 0^-} e^{\frac{1}{x}} = \lim\limits_{u \to -\infty} e^u = 0$

Review: Sec 1.8 in textbook "removable discontinuity"

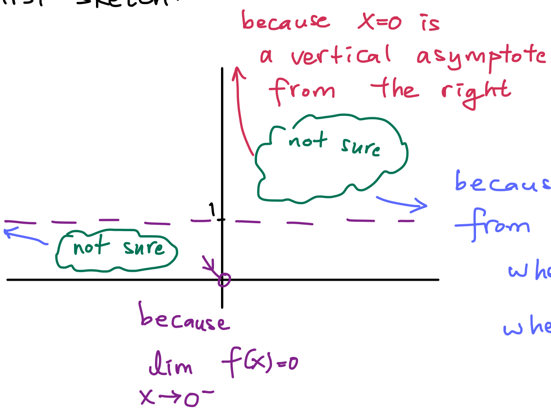
* $\lim\limits_{x \to \infty} e^{\frac{1}{x}} = ?$ As $x \to \infty$, $\frac{1}{x} \to 0$

So $\lim\limits_{x \to \infty} e^{\frac{1}{x}} = e^{\left(\lim\limits_{x \to \infty} \frac{1}{x}\right)} = e^0 = 1$

* $\lim\limits_{x \to -\infty} e^{\frac{1}{x}} = e^{\left(\lim\limits_{x \to -\infty} \frac{1}{x}\right)} = e^0 = 1$

So $y=1$ is a horizontal asymptote to the left and right

First sketch:



because $y=1$ is a horizontal asymptote from left and right &

when x is positive, $e^{\frac{1}{x}}$ is bigger than 1

when x is negative, $e^{\frac{1}{x}}$ is smaller than 1

To check when $f(x)$ is increasing/decreasing/concave up or down, compute $f'(x)$ and $f''(x)$:

$$f(x) = e^{\frac{1}{x}}$$

$$f'(x) = e^{\boxed{\frac{1}{x}}} \frac{d}{dx}\left(\boxed{\frac{1}{x}}\right) = e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right) = \underbrace{-x^{-2}}_{\substack{\text{always}\\ \text{negative}}} \underbrace{e^{\frac{1}{x}}}_{\text{always positive}}$$

Chain Rule
Review Sec 2.5

because $x^2>0$ when $x\neq 0$

so $f(x)$ is decreasing everywhere

$$f''(x) = -\left[x^{-2}\left(e^{\frac{1}{x}}\right)' + -2x^{-3}\cdot e^{\frac{1}{x}}\right] = -\left[x^{-2}\overbrace{\left(-x^{-2}e^{\frac{1}{x}}\right)}^{\left(e^{\frac{1}{x}}\right)'} - 2x^{-3}\cdot e^{\frac{1}{x}}\right]$$

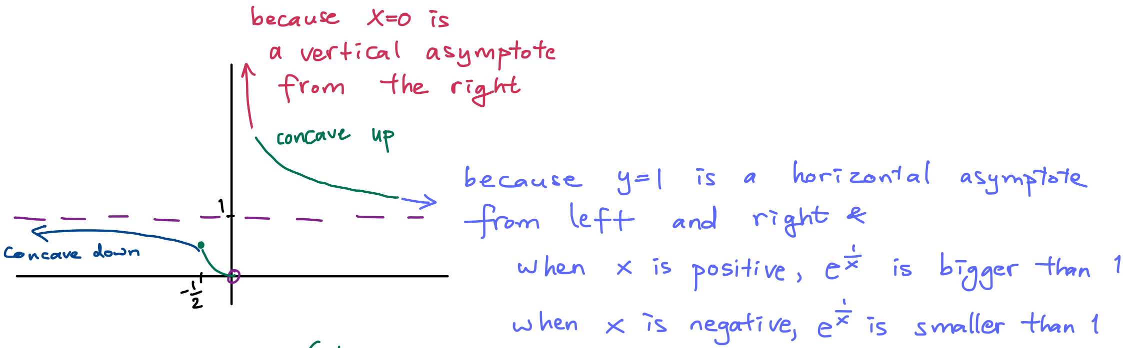
Product Rule
Review Sec 2.3

$$= x^{-4}e^{\frac{1}{x}} + 2x^{-3}e^{\frac{1}{x}} = \left(\frac{1}{x^4}+\frac{2}{x^3}\right)e^{\frac{1}{x}} = \frac{(1+2x)}{\underbrace{x^4}_{\substack{\text{always}\\ \text{positive}}}}\underbrace{e^{\frac{1}{x}}}_{\substack{\text{always}\\ \text{positive}}}$$

So $f''(x)$ is positive when $1+2x>0 \iff 2x>-1$ and $x\neq 0$ $\iff x>-\frac{1}{2}$ and $x\neq 0$
(concave up)

$f''(x)$ is negative when $1+2x<0 \iff 2x<-1 \iff x<-\frac{1}{2}$
(concave down)

Final sketch:



concave up on $\left(-\frac{1}{2}, 0\right)$

and $(0, \infty)$

Concave down on $\left(-\infty, -\frac{1}{2}\right)$

— End of Example 1 —

Fact: Since $\frac{d}{dx} e^x = e^x$, we have $\boxed{\int e^x \, dx = e^x + C}$

Ex 2 $\quad \int e^{(x^3)} x^2 \, dx = ?$ $\quad$ Review u-substitution Sec 4.5

Try $\boxed{u = x^3}$, $du = 3x^2 \, dx$

$\boxed{\frac{1}{3} du = x^2 \, dx}$

$$\int e^{\boxed{(x^3)}} \boxed{x^2 \, dx} = \int e^{\boxed{u}} \boxed{\frac{1}{3} du} = \frac{1}{3} \int e^u \, du = \frac{1}{3} e^u + C$$

$$= \boxed{\frac{1}{3} e^{(x^3)} + C}$$

Ex 2

— the end —