Ch 6 main idea

- Inverse functions
- exponential $\overset{\text{inverse}}{\dashleftarrow\dashrightarrow}$ logarithmic
- Inverse trig functions

Sec 6.1 Inverse functions

Def • A function $g: \underbrace{A}_{\substack{\text{all possible} \\ \text{inputs} \\ \text{(domain)}}} \longrightarrow \underbrace{B}_{\substack{\text{all possible} \\ \text{outputs}}}$ is called one-to-one (1-1) if...

(image in most textbooks – range in ours)

if $a_1 \neq a_2$ then $g(a_1) \neq g(a_2)$

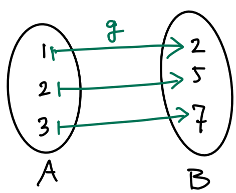
"g never takes on the same value twice"

• If you can find two inputs a_1, a_2 where $g(a_1) = g(a_2)$,

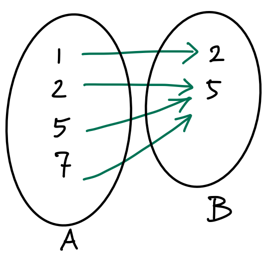
we say g is not one-to-one.

"g takes on the same value twice"

Ex 1



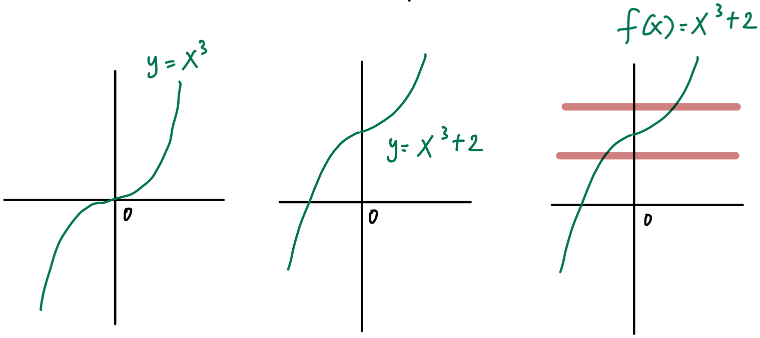
is one-to-one



is not one-to-one

Ex 2 Let $f(x) = x^3 + 2$. Do you think $f(x)$ is one-to-one?

Solution 1: Sketch $f(x) = x^3 + 2$



Horizontal line test:

- If no horizontal line intersects the graph $y = f(x)$ twice, then $f(x)$ is one-to-one.
- If a horizontal line intersects the graph $y = f(x)$ twice, then $f(x)$ is one-to-one.

Solution 2: (From prev page def: f is 1-1 means if $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$)

If $a_1 \neq a_2$ then $a_1^3 \neq a_2^3$ so $f(a_1) = a_1^3 + 2 \neq a_2^3 + 2 = f(a_2)$

Def If $g : A \longrightarrow B$ is one-to-one,

all possible inputs (domain of g) — all possible outputs of g (image of g)

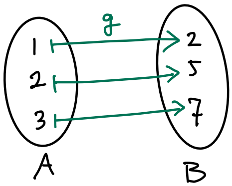
we can define the inverse function of g

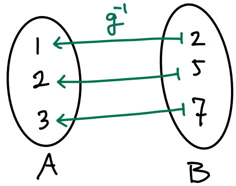
Read "g inverse" $g^{-1} : B \longrightarrow A$

by $g^{-1}(b) = a$ if $g(a) = b$

"If g sends a to b, then g^{-1} sends b back to a"

From Ex 1





Simply reverse the arrows

$g(1) = 2$ means $g^{-1}(2) = 1$

$g(2) = 5$ means $g^{-1}(5) = 2$

$g(3) = 7$ means $g^{-1}(7) = 3$.

Facts * domain of g = image of g^{-1}

image of g = domain of g^{-1}

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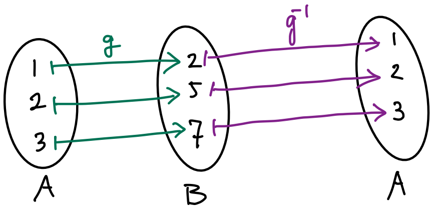
* $g^{-1}(g(x)) = x$ for all x in domain of g



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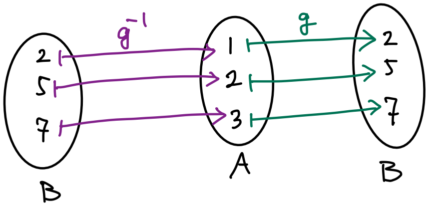
* $g(g^{-1}(x)) = x$ for all x in domain of g^{-1}



How to ... find $f^{-1}(x)$
graph $f^{-1}(x)$
of a one-to-one function $f(x)$.

Same idea for both Use the fact $f^{-1}(x) = y \iff f(y) = x$

How to find $f^{-1}(x)$:

Step 1. Write $y = f(x)$

Step 2. Solve for x (if possible)

Step 3. Interchange x & y to get $y = f^{-1}(x)$

(From Ex 2) $f(x) = x^3 + 2$

Step 1. Write $y = x^3 + 2$

Step 2. $y - 2 = x^3$

$\sqrt[3]{y-2} = x$

Step 3. $y = \sqrt[3]{x-2}$

$f^{-1}(x) = \sqrt[3]{x-2}$

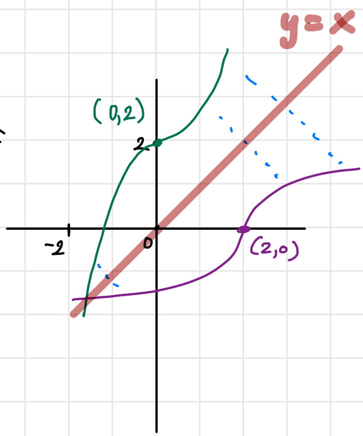
The inverse function of $f(x) = x^3 + 2$ is $f^{-1}(x) = \sqrt[3]{x-2}$

How to graph f^{-1}:

- $f(a) = b$ iff $f^{-1}(b) = a$ means

point (a,b) is on the graph of f iff (b,a) is on the graph of f^{-1}

- To get from (a,b) to (b,a), we reflect about the line $y = x$



To get the graph of f^{-1}, reflect the graph of f about the line $y = x$

Q: Is $f^{-1}(x)$ the same as $\frac{1}{f(x)} = (f(x))^{-1}$?

In general, no!

Thm If f is a one-to-one function which is continuous, then its inverse function f^{-1} is also continuous

Idea
- Since f is continuous, graph of f has no break (it consists of one connected piece).
- The graph of f^{-1} is a reflection of the graph of f, so the graph of f^{-1} also has no break.
- So f^{-1} is also a continuous function.

Thm If f is a one-to-one function which is differentiable, then its inverse function f^{-1} is also differentiable, except when the tangent line is vertical.

How to find the derivative of f^{-1} using implicit differentiation, by example

Ex 3

$f(x) = 2x + \cos x$. Find $(f^{-1})'(1)$

Note: $f(x)$ is one-to-one because
$f'(x) = 2 - \sin x > 0$ (since $\sin x > 0$)
so $f(x)$ is increasing
(hence no horizontal line can cross the graph twice)

end of Day 1

$$y = f^{-1}(x)$$

Apply f to both sides:

$$f(y) = f(f^{-1}(x))$$

$$f(y) = x$$

$$2y + \cos y = x$$

Differentiate with respect to x:

$$\frac{d}{dy}(2y + \cos y)\frac{dy}{dx} = \frac{d}{dx}x$$

$$(2 - \sin y)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{2 - \sin y}$$

Substitute $y = f^{-1}(x)$:

$$\frac{dy}{dx} = \frac{1}{2 - \sin(f^{-1}(x))}$$

So $(f^{-1})'(x) = \dfrac{1}{2 - \sin(f^{-1}(x))}$

The derivative of f^{-1} at 1 is $\frac{1}{2}$

$$(f^{-1})'(1) = \frac{1}{2 - \sin(f^{-1}(1))} = \frac{1}{2 - \sin(0)} = \boxed{\frac{1}{2}}$$

$f^{-1}(1) = \boxed{?} \iff f(\boxed{?}) = 1$

$2\boxed{?} + \cos\boxed{?} = 1$

$\boxed{?} = 0$ works

so $f^{-1}(1) = 0$