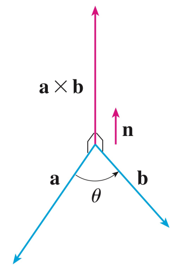
Goal

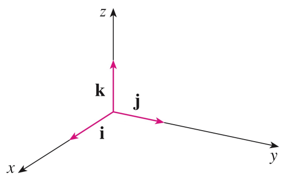
Given two nonzero vectors $\vec{a}$ and $\vec{b}$ in 3D,

find a third nonzero vector $\vec{c}$ that is

perpendicular to both $\vec{a}$ and $\vec{b}$.

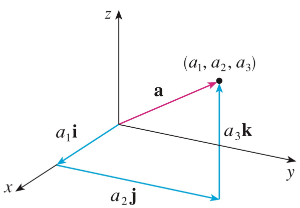


standard basis vectors



$$\mathbf{i} = \langle 1, 0, 0 \rangle \qquad \mathbf{j} = \langle 0, 1, 0 \rangle \qquad \mathbf{k} = \langle 0, 0, 1 \rangle$$

length is 1, direction : positive axis



If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then we can write

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle$$

$$= a_1\langle 1, 0, 0 \rangle + a_2\langle 0, 1, 0 \rangle + a_3\langle 0, 0, 1 \rangle$$

2 $$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

Thus any vector in V_3 can be expressed in terms of **i**, **j**, and **k**. For instance,

$$\langle 1, -2, 6 \rangle = \mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$$

A **determinant of order 2** is defined by $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

E.g. $\begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} = 1(7) - 3(2) = 1$

Def of cross product

Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

Write $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ just a symbol to help us memorize the def

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Def

$$\boxed{\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}}$$

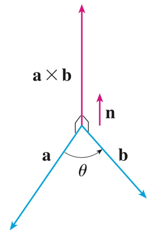
EXAMPLE 1 If $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, 7, -5 \rangle$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \mathbf{k}$$

$$= (-15 - 28)\,\mathbf{i} - (-5 - 8)\,\mathbf{j} + (7 - 6)\,\mathbf{k} = -43\mathbf{i} + 13\mathbf{j} + \mathbf{k}$$

8 Theorem The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.



ⓘ show that $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k} \right) \cdot \langle a_1, a_2, a_3 \rangle$$

$$\left\langle \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right\rangle \cdot \langle a_1, a_2, a_3 \rangle$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3$$

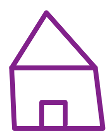
$$= a_1(a_2b_3 - a_3b_2) - a_2(a_1b_3 - a_3b_1) + a_3(a_1b_2 - a_2b_1)$$

$$= a_1a_2b_3 - a_1b_2a_3 - a_1a_2b_3 + b_1a_2a_3 + a_1b_2a_3 - b_1a_2a_3$$

$$= 0$$

(Test your understanding)

ⓘⓘ Show that $\vec{a} \times \vec{b}$ is orthogonal to $\vec{b}$ by computing $(\vec{a} \times \vec{b}) \cdot \vec{b} \overset{?}{=} 0$.



Geometric def of cross product

If θ is the angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ (so $0 \leqslant \theta \leqslant \pi$), then

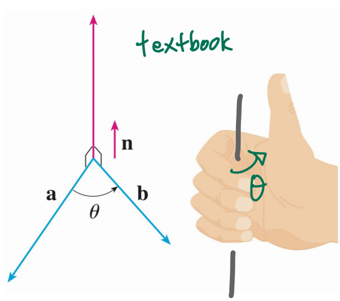
$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \underbrace{|\vec{\mathbf{a}}||\vec{\mathbf{b}}| \sin\theta}_{\text{scalar}} \underbrace{\vec{n}}_{\text{unit vector}}$$

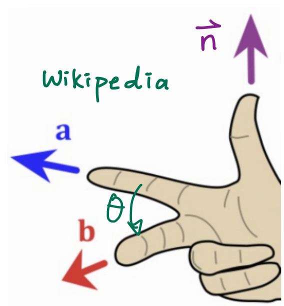
where $\vec{n}$ is ...

(i) a unit vector (meaning $|\vec{n}| = 1$)

(ii) perpendicular to both $\vec{a}$ and $\vec{b}$

(iii) the direction of $\vec{n}$ is given by the right-hand rule





Pick your favorite method & stick with it!

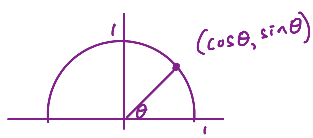
Fact

Two nonzero vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are parallel if and only if

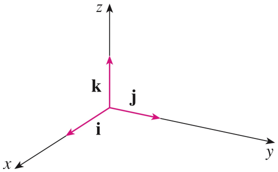
$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{0}}$$

Why?

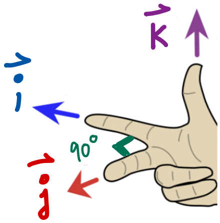
$\sin\theta$ is 0 if and only if $\theta = 0$ or π



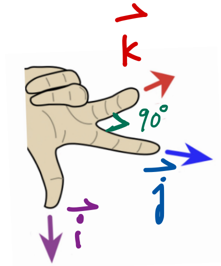
standard basis vectors



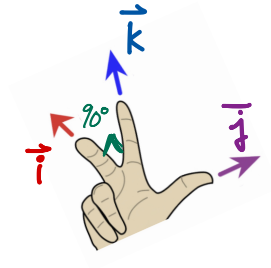
$$\mathbf{i} = \langle 1, 0, 0 \rangle \qquad \mathbf{j} = \langle 0, 1, 0 \rangle \qquad \mathbf{k} = \langle 0, 0, 1 \rangle$$



$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$



$$\mathbf{j} \times \mathbf{k} = \mathbf{i}$$



$$\mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\vec{j} \times \vec{i} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \hat{i} - \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \vec{k}$$

$$= -\vec{k}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

check check

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11 Properties of the Cross Product

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$ for all vectors $\vec{a}, \vec{b}$

In general,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$ Not associative!

E.g.

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}$$

Webwork Problem 5

If $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, 7, -5 \rangle$,

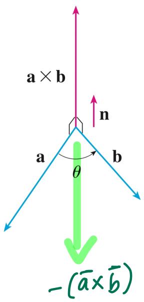
find a unit vector with positive first coordinate orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

Solution

i $\mathbf{a} \times \mathbf{b}$

$$= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \mathbf{k}$$

$$= (-15 - 28)\,\mathbf{i} - (-5 - 8)\,\mathbf{j} + (7 - 6)\,\mathbf{k} = -43\mathbf{i} + 13\mathbf{j} + \mathbf{k}$$



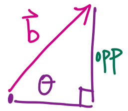
ii To get a vector in opposite direction as $\vec{a} \times \vec{b}$, do scalar multiplication

$$-1\left(-43\mathbf{i} + 13\mathbf{j} + \mathbf{k}\right) = \langle 43, -13, -1 \rangle$$

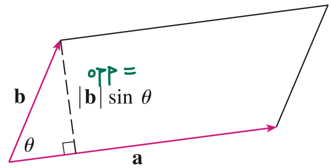
iii To get a unit vector, scalar multiply

$$\frac{1}{l}\langle 43, -13, -1 \rangle \quad \text{where } l = \underbrace{\sqrt{(43)^2 + (-13)^2 + (-1)^2}}_{\text{length of } \vec{a}\times\vec{b}}$$

Physical meaning of $|\vec{a} \times \vec{b}|$



$\sin\theta = \dfrac{opp}{|\vec{b}|} \Rightarrow$



Area of parallelogram is (height)$\cdot$(base)

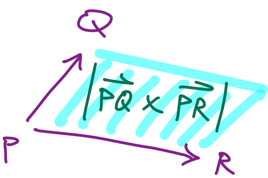
$= |\vec{b}| \sin\theta \cdot |\vec{a}|$

Since $\vec{a} \times \vec{b} = \underbrace{|\vec{a}|\,|\vec{b}| \sin\theta}_{\text{length of } \vec{a}\times\vec{b}} \; \underbrace{\vec{n}}_{\text{unit vector}}$, we have ...

> The length of the cross product **a** × **b** is equal to the area of the parallelogram determined by **a** and **b**.

Webwork 6,7

EXAMPLE 4 Find the area of the triangle with vertices $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.



$$\overrightarrow{PQ} = (-2 - 1)\,\mathbf{i} + (5 - 4)\,\mathbf{j} + (-1 - 6)\,\mathbf{k} = -3\mathbf{i} + \mathbf{j} - 7\mathbf{k}$$

$$\overrightarrow{PR} = (1 - 1)\,\mathbf{i} + (-1 - 4)\,\mathbf{j} + (1 - 6)\,\mathbf{k} = -5\mathbf{j} - 5\mathbf{k}$$

We compute the cross product of these vectors:

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix}$$

$$= (-5 - 35)\,\mathbf{i} - (15 - 0)\,\mathbf{j} + (15 - 0)\,\mathbf{k} = -40\mathbf{i} - 15\mathbf{j} + 15\mathbf{k}$$

Area of parallelogram is $|\overrightarrow{PQ} \times \overrightarrow{PR}| \overset{\text{length formula}}{=} \sqrt{(-40)^2 + (-15)^2 + 15^2} = 5\sqrt{82}$

The area A of the triangle PQR is half the area of this parallelogram, that is, $\frac{5}{2}\sqrt{82}$. ■

Proof that the first definition and the geometric definition are equivalent

9 Theorem If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \leqslant \theta \leqslant \pi$), then

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin \theta$$

PROOF From the definitions of the cross product and length of a vector, we have

$$\begin{aligned}
|\mathbf{a} \times \mathbf{b}|^2 &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\
&= a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2 + a_3^2b_1^2 - 2a_1a_3b_1b_3 + a_1^2b_3^2 \\
&\qquad + a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2 \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\
&= |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\
&= |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta \qquad \text{(by Theorem 12.3.3)} \\
&= |\mathbf{a}|^2|\mathbf{b}|^2(1 - \cos^2\theta) \\
&= |\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta
\end{aligned}$$

Taking square roots and observing that $\sqrt{\sin^2\theta} = \sin\theta$ because $\sin\theta \geqslant 0$ when $0 \leqslant \theta \leqslant \pi$, we have

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

■

Since a vector is completely determined by its magnitude and direction, we can now say that $\mathbf{a} \times \mathbf{b}$ is the vector that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, whose orientation is determined by the right-hand rule, and whose length is $|\mathbf{a}||\mathbf{b}|\sin\theta$. In fact, that is exactly how physicists *define* $\mathbf{a} \times \mathbf{b}$.