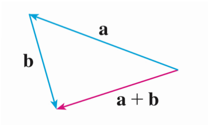
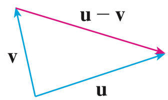
Recall Sec 12.2

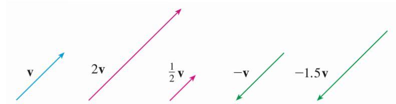
- Sum of two vectors



- Difference of two vectors



- multiplying a vector by a number / scalar



- Can we "multiply" two vectors ?

dot product : " product" of two vectors is a number

cross product : " product" of two vectors is a vector

12.3 The Dot Product (also called the scalar product or inner product)

1 Definition

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 .$$

If $\vec{a} = \langle a_1, a_2 \rangle$ and $\vec{b} = \langle b_1, b_2 \rangle$, then $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$

Note $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ (The order of the vectors doesn't matter)

EXAMPLE

$$\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle = 2(3) + 4(-1) = 2$$

$$\langle -1, 7, 4 \rangle \cdot \langle 6, 2, -\tfrac{1}{2} \rangle = (-1)(6) + 7(2) + 4(-\tfrac{1}{2}) = 6$$

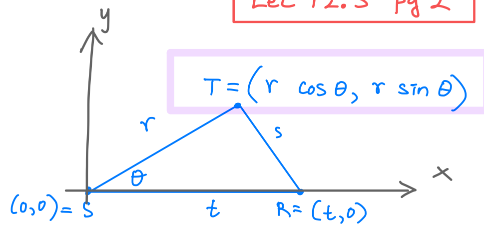
Fact

$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$$

$$\vec{a} \cdot \vec{a} \overset{def}{=} \langle a_1, a_2 \rangle \cdot \langle a_1, a_2 \rangle = a_1^2 + a_2^2$$

$$|\vec{a}|^2 = (\text{length of } \vec{a})^2 = \left(\sqrt{a_1^2 + a_2^2}\right)^2 = a_1^2 + a_2^2$$

Law of Cosines



S = Distance between R and T

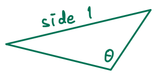
$$= \sqrt{(t - r\cos\theta)^2 + (0 - r\sin\theta)^2}$$

$$s^2 = (t - r\cos\theta)^2 + r^2(\sin\theta)^2$$

$$s^2 = t^2 - 2tr\cos\theta + r^2(\cos\theta)^2 + r^2(\sin\theta)^2$$

$$s^2 = t^2 + r^2 - 2tr\cos\theta$$ (since $\cos^2\theta + \sin^2\theta = 1$)

"Law of Cosines"



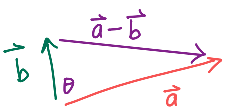
$$(\text{side } 1)^2 = (\text{side } 2)^2 + (\text{side } 3)^2 - 2(\text{side } 1)(\text{side } 2)\cos\theta$$

Geometric meaning of the dot product

Let $\vec{a} = \langle a_1, a_2 \rangle$

$\vec{b} = \langle b_1, b_2 \rangle$

$\vec{a} - \vec{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$



By law of cosines,

① $|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$

② $|\vec{a} - \vec{b}|^2 \overset{\text{Length formula}}{=} (a_1 - b_1)^2 + (a_2 - b_2)^2$

$$= a_1^2 - 2a_1b_1 + b_1^2 + a_2^2 - 2a_2b_2 + b_2^2$$

$$= \underbrace{a_1^2 + a_2^2} + \underbrace{b_1^2 + b_2^2} - 2\underbrace{(a_1b_1 + a_2b_2)}_{\text{dot product}}$$

$$= |\vec{a}|^2 + |\vec{b}|^2 - 2\,\vec{a}\cdot\vec{b}$$

① $|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$

② $|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\,\vec{a}\cdot\vec{b}$ $-$

$$0 = -2|\vec{a}||\vec{b}|\cos\theta + 2\,\vec{a}\cdot\vec{b}$$

So $$\boxed{\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta}$$

Geometric meaning of the dot product

3 Theorem If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

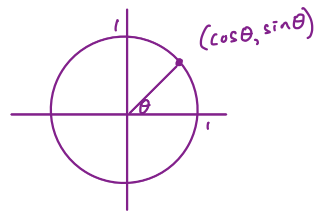
6 Corollary If θ is the angle between the nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

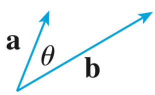
$\cos(\theta)$ is ... positive when $0 \leq \theta < \frac{\pi}{2}$

0 when $\theta = \frac{\pi}{2}$

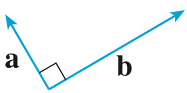
negative when $\frac{\pi}{2} < \theta \leq \pi$



So we can determine acute/right/obtuse angle by computing $\vec{a} \cdot \vec{b}$

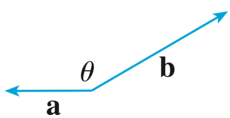


$\mathbf{a} \cdot \mathbf{b} > 0$
θ acute



$\mathbf{a} \cdot \mathbf{b} = 0$
$\theta = \pi/2$

Two vectors $\vec{a}, \vec{b}$ are orthogonal or perpendicular if $\vec{a} \cdot \vec{b} = 0$



$\mathbf{a} \cdot \mathbf{b} < 0$
θ obtuse

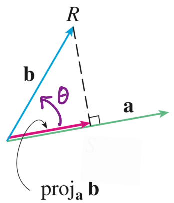
Webwork Prob 1

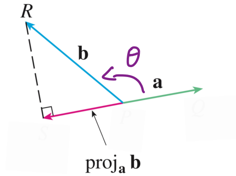
$\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle = \quad 2(3) + 4(-1) = 2$

so the angle between $\langle 2,4 \rangle$ and $\langle 3,-1 \rangle$ is acute

Projections

The <u>vector projection</u> of $\vec{b}$ onto $\vec{a}$ is the vector $\text{proj}_{\vec{a}}\vec{b}$





when the angle θ between $\vec{a}$ and $\vec{b}$ is acute (smaller than $\frac{\pi}{2}$)

when the angle θ between $\vec{a}$ and $\vec{b}$ is obtuse (bigger than $\frac{\pi}{2}$)

Think of this vector $\text{proj}_{\vec{a}}\vec{b}$ as a shadow of $\vec{b}$

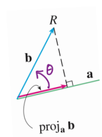
The <u>scalar projection</u> of $\vec{b}$ onto $\vec{a}$ (denoted $\text{comp}_{\vec{a}}\vec{b}$)

or <u>component of $\vec{b}$ along $\vec{a}$</u>

is the length of $\text{proj}_{\vec{a}}\vec{b}$ if θ is less than $\frac{\pi}{2}$

(-1) times length of $\text{proj}_{\vec{a}}\vec{b}$ if θ is bigger than $\frac{\pi}{2}$

Scalar projection of **b** onto **a**: $\boxed{\text{comp}_{\mathbf{a}}\, \mathbf{b} = |b| \cos\theta}$ why?



$\cos\theta = \frac{\text{adj}}{\text{hyp}}$

Since $\vec{a}\cdot\vec{b} = |a||b|\cos\theta$,

we have $\boxed{\text{comp}_{\mathbf{a}}\, \mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}}$

$\text{comp}_{\vec{a}}\vec{b}$ is a number

Vector projection of **b** onto **a**: $\text{proj}_{\mathbf{a}}\, \mathbf{b} = \underbrace{\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)}_{(\pm 1)\text{ length } \text{comp}_{\vec{a}}\vec{b}} \underbrace{\frac{\mathbf{a}}{|\mathbf{a}|}}_{\text{unit vector in the direction of } \vec{a}} = |\vec{b}|\cos\theta\ \frac{\vec{a}}{|\vec{a}|}$

$\text{proj}_{\vec{a}}\vec{b}$ is a vector