Lecture 11.9 Representations of functions as power series

We've seen examples of convergent power series—but can we write an explicit function that is represented by a power series?

Consider $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$

This is geometric series with ratio $= x$

The power series converges if $|x| < 1$

so the interval of convergence is $(-1, 1)$

When x is in the interval of convergence, $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$

and $\sum_{n=5}^{\infty} x^n = x^5 + x^6 + x^7 + \ldots = x^5(1 + x + x^2 + \ldots) = x^5 \frac{1}{1-x}$

Extending this idea

For $|x| < 1$, we can express $\frac{1}{1-x}$ as the power series $1 + x + x^2 + \cdots$

Can we express $\frac{1}{3-x}$ as a power series? What values of x would work?

$$\frac{\frac{1}{3}\,1}{\frac{1}{3}(3-x)} = \frac{\frac{1}{3}}{(1-\frac{x}{3})}$$

$$= \frac{1}{3}\,\frac{1}{1-(\frac{x}{3})}$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n \quad \text{if } \left|\frac{x}{3}\right| < 1 \iff |x| < 3$$

radius of convergence

Interval of convergence is $(-3, 3)$

because $\sum\left(\frac{x}{3}\right)^n = \sum\left(\frac{1}{3}\right)^n(x-0)^n$

Question: What is the center? A. 0 B. 1 C. 2 D. 3

Question: What is the radius of convergence? A. 1 B. 3 C. 1/3

(Webwork Problem 2)

Find a power series representation for $f(x) = \dfrac{5}{1+4x^2}$ and find its interval of convergence.

$$\frac{5}{1+4x^2} = 5\,\frac{1}{1-(-4x^2)}$$

$$= 5\sum_{n=0}^{\infty}(-4x^2)^n \quad \text{if } |-4x^2| < 1 \iff |x^2| < \tfrac{1}{4}$$

$$\iff |x| < \underbrace{\tfrac{1}{2}}_{\text{radius of convergence}}$$

$$= 5\sum_{n=0}^{\infty}(-1)^n 4^n x^{2n}$$

Interval of convergence:

$\left(-\frac{1}{2}, \frac{1}{2}\right)$

Question: What is the radius of convergence?

A. 1 B. 2 C. 1/2 D. 4 E. 1/4

$\boxed{\text{A similar problem}}$

Similar to

(Webwork Problem 3)

Find a power series representation for $f(x) = \dfrac{2x^4}{2-3x}$ and find its interval of convergence.

$$\frac{\frac{1}{2}\,2x^4}{\frac{1}{2}(2-3x)} = \frac{x^4}{1-\frac{3}{2}x}$$

$$= x^4\,\frac{1}{1-\left(\frac{3}{2}x\right)}$$

Radius of convergence

$$= x^4\sum_{n=0}^{\infty}\left(\frac{3}{2}x\right)^n \quad \text{if } \left|\frac{3}{2}x\right| < 1 \iff |x| < \frac{2}{3}$$

$$= x^4\sum_{n=0}^{\infty}\left(\frac{3}{2}\right)^n x^n$$

$$= \sum_{n=0}^{\infty}\left(\frac{3}{2}\right)^n x^{4+n}$$

Interval of convergence : $\left(-\frac{2}{3}, \frac{2}{3}\right)$

Similar to
(Webwork Problem 11)

What happens if we find an antiderivative for the equation below?

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$$ for $|x| < 1$

$$\int \frac{1}{1+x}\,dx = \int\left(\sum_{n=0}^{\infty}(-1)^n x^n\right)dx$$

$$= \int \left(1 - x + x^2 - x^3 + \ldots\right) dx$$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) + C$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$$

Differentiation and Integration

We can use differentiation and integration to express other kinds of functions as powers series:

Theorem: If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) \overset{def}{=} c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

$(c_n x^n)' = n c_n x^{n-1}$

(I) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

term-by-term differentiation & term-by-term integration

$\int c_n x^n\,dx = c_n \frac{x^{n+1}}{n+1} + C$

(II) $\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots =$
$C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

The radii of convergence for both of these power series is R.

Radius of convergence stays the same (after term-by-term differentiation and integration)

(Webwork Problem 5)

Find a power series representation (centered at 0) for $f(x) = \dfrac{1}{(5+x)^2}$.

Step 0 $\quad \dfrac{d}{dx}\left[\dfrac{1}{(5+x)}\right] = -\dfrac{1}{(5+x)^2}$

$$\frac{d}{dx}\left[-\frac{1}{5+x}\right] = \frac{1}{(5+x)^2}$$

Step 1 $\quad -\dfrac{1}{5+x} = -\dfrac{\frac{1}{5}}{\frac{1}{5}(5+x)}$

$$= -\frac{1}{5}\,\frac{1}{1-\left(-\frac{x}{5}\right)}$$

$$= -\frac{1}{5}\sum_{n=0}^{\infty}\left(-\frac{x}{5}\right)^n \quad \text{for } \left|-\frac{x}{5}\right| < 1 \iff |x| < 5$$

Radius of convergence: 5

$$= -\frac{1}{5}\sum_{n=0}^{\infty}\left(-\frac{1}{5}\right)^n x^n$$

$$= \sum_{n=0}^{\infty}\left(-\frac{1}{5}\right)^{n+1} x^n$$

Step 2 $\quad \dfrac{1}{(5+x)^2} = \dfrac{d}{dx}\left[-\dfrac{1}{5+x}\right]$

$$= \frac{d}{dx}\left[\sum_{n=0}^{\infty}\left(-\frac{1}{5}\right)^{n+1} x^n\right]$$

$$= \frac{d}{dx}\left[\left(-\frac{1}{5}\right)x^0 + \sum_{n=1}^{\infty}\left(-\frac{1}{5}\right)^{n+1} x^n\right]$$

$$= \boxed{0 + \sum_{n=1}^{\infty}\left(-\frac{1}{5}\right)^{n+1} n\,x^{n-1}}$$

by Thm (term-by-term differentiation)

(you can stop here, but webwork asks for $\sum_{n=0}^{\infty} \boxed{?}$, so let's keep going)

$$= \sum_{n=0}^{\infty}\left(-\frac{1}{5}\right)^{n+2}(n+1)\,x^n$$

Example

Find $\int \ln(1+t^4)\,dt$ as a power series, and find its radius of convergence.

Step 1:

See Textbook Example 6 pg 795

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int\left(\sum_{n=0}^{\infty}(-1)^n x^n\right)dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} + C$$

from page 5

To find C, plug in the center $x=0$ of the power series:

$$\ln(1+0) = 0 + C$$

so $C = 0$

Here, we have to solve for C.

$$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} \quad \text{for } |x| < 1$$

Step 2:

So
$$\ln(1+t^4) = \sum_{n=0}^{\infty}(-1)^n \frac{(t^4)^{n+1}}{n+1} \quad \text{for } |t^4| < 1 \iff |t| < 1$$

Radius of convergence is 1

$$= \sum_{n=0}^{\infty}(-1)^n \frac{t^{4n+4}}{n+1}$$

Step 3:

$$\int \ln(1+t^4)\,dt = \sum_{n=0}^{\infty}(-1)^n \frac{t^{4n+5}}{(n+1)(4n+5)} + \text{Constant}$$

Same radius of convergence, 1

Copy Textbook 11.9 Example 7, pg 796

Find $\arctan(x)$ as a power series, and find its radius of convergence. Do after class

EXAMPLE 7 Find a power series representation for $f(x) = \tan^{-1}x$.

SOLUTION We observe that $f'(x) = 1/(1+x^2)$ and find the required series by integrating the power series for $1/(1+x^2)$ found in Example 1.

$$\tan^{-1}x = \int \frac{1}{1+x^2}\,dx = \int (1 - x^2 + x^4 - x^6 + \cdots)\,dx$$
$$= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

To find C we put $x = 0$ and obtain $C = \tan^{-1}0 = 0$. Therefore

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$
$$= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

Since the radius of convergence of the series for $1/(1+x^2)$ is 1, the radius of convergence of this series for $\tan^{-1}x$ is also 1.

Answer: $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$ for $|x| < 1$

Text Ex 7

(WEBWORK PROBLEM 9)

Use the fact $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ (with radius of convergence 1) to find a power series representation of $\int \frac{\arctan(2x)}{x}\,dx$. ← This cannot be solved by Chapter 7 techniques

Find its radius of convergence.

$$\arctan(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{2n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{2n+1}}{2n+1} \quad \text{for} \quad |2x| < 1 \iff |x| < \frac{1}{2}$$

$$\frac{\arctan(2x)}{x} = \frac{1}{x}\sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{2n+1}}{2n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1}}{2n+1} x^{2n}$$

$$\int \frac{\arctan(2x)}{x}\,dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1}}{2n+1} x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1}}{2n+1} \frac{x^{2n+1}}{2n+1} + C$$

$$= \boxed{\left(2x - \frac{2^3x^3}{9} + \frac{2^5x^5}{25} - \frac{2^7x^7}{49} + \dots\right) + C} \checkmark$$

Radius of convergence is the same as for the series for arctan(2x) : $\frac{1}{2}$

(WEBWORK PROBLEM 10)

Find $\dfrac{4x+5}{5x^2-19x-4}$ as a power series, and find its radius of convergence.

Partial fraction :

$$\frac{4x+5}{(5x+1)(x-4)} = \frac{A}{5x+1} + \frac{B}{x-4}$$

$$4x+5 = A(x-4) + B(5x+1)$$

$x=4$: $16+5 = B(21) \Rightarrow B=1$

$x=-\frac{1}{5}$: $-\frac{4}{5}+5 = A\left(-\frac{1}{5}-4\right)$

$$\frac{21}{5} = A\left(\frac{-21}{5}\right) \Rightarrow A=-1$$

$$f(x) = -\frac{1}{5x+1} + \frac{1}{x-4}$$

$$= -\frac{1}{1-(-5x)} - \frac{1}{4-x}$$

$$= -\sum_{n=0}^{\infty}(-5x)^n - \left(\frac{1}{4\left(1-\frac{x}{4}\right)}\right)$$

$$= \sum_{n=0}^{\infty}(-1)(-5)^n x^n - \sum_{n=0}^{\infty}\frac{1}{4}\left(\frac{1}{4}\right)^n x^n$$

$$= \boxed{\sum_{n=0}^{\infty}\left[(-1)(-5)^n - \frac{1}{4}\left(\frac{1}{4}\right)^n\right] x^n}$$

Interval of convergence is the intersection of the interval for $\sum_{n=0}^{\infty}(-5x)^n$ which is $\left(-\frac{1}{5}, \frac{1}{5}\right)$ and the interval for $\sum_{n=0}^{\infty}\frac{1}{4}\left(\frac{x}{4}\right)^n$ which is $(-4, 4)$

So it is $\boxed{\left(-\frac{1}{5}, \frac{1}{5}\right)}$